Generic Size Theory

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1. Introduction – The Euclidian Paradigm

As mentioned in the previous chapter, Euclid's *Elements* provides a paradigm for what a theory should be. Recall that the Euclidean paradigm may be summarized as follows.

(1) a *given* realm of phenomena (the **data**)

(2) a set of **laws**/generalizations gleaned from (1)

(3) a *proposed* **theory** to explain, systematize, organize, and/or codify (2), which includes:

  (a) a proposed set of primitive concepts
  (b) a proposed set of primitive postulates (axioms)
  (c) a proposed set of common notions (logical and notational machinery)
  (d) a proposed set of definitions
  (e) a demonstration of how (a)-(d) codifies (2)

2. A Simple Example – A Generic Theory of Size

1. Generic Size

By way of illustrating the Euclidean paradigm in a small carefully-confined environment, we consider a simple theory – which we call 'Generic Size Theory' (GST). In particular, the notion of size we are considering is generic, and includes what is common to all specific notions of size. In particular we can measure and compare the sizes of things based on many different criteria. Consider the following examples.

India is bigger than Australia (in population)
Australia is bigger than India (in area)
The Pacific Ocean is bigger than the Atlantic Ocean (in area, and in volume)
The Eiffel Tower is bigger than the Great Pyramid (in height)[1]
The Great Pyramid is bigger than the Eiffel Tower (in weight)
Hulk Hogan is bigger than Paul Hogan (in height and weight)
A year is bigger than a month (in length of time)
Plato is bigger than Thales (in "stature")
Thanksgiving is bigger than Ground Hog Day (in "importance")
Killing a person is bigger than killing a fly (in moral seriousness)

We are considering what can be said about "size in general". Also, please bear in mind that the proposed theory is not intended to say *everything* there is to say about size, but only to systematize the most prominent "laws of size".

2. Laws of Size

The laws of size, generically understood, include the following among others.

(1) if $x > y$, and $y > z$, then $x > z$
if $x < y$, and $y < z$, then $x < z$

(2) if $x \ngtr y$, and $y \ngtr z$, then $x \ngtr z$
if $x \nless y$, and $y \nless z$, then $x \nless z$

(3) if $x \geqslant y$, and $y \geqslant z$, then $x \geqslant z$
if $x \leqslant y$, and $y \leqslant z$, then $x \leqslant z$

(4) if $x > y$, then $y \ngtr x$
if $x < y$, then $y \nless x$

(5) $x \ngtr x$
$x \nless x$

(6) $x \leqslant x$
$x \geqslant x$

(7) $x \equiv x$

(8) if $x \equiv y$, then $y \equiv x$

(9) if $x \equiv y$, and $y \equiv z$, then $x \equiv z$

(10) if $x > y$, and $y \equiv z$, then $x > z$

(11) if $x \equiv y$, and $y > z$, then $x > z$

(12) $x > y$, or $y > x$, or $x \equiv y$

These statements involve two logical devices, one fairly trivial, the other fairly profound.

(1) (symbolic) abbreviations of key concepts;

For example, in this context, the special symbol '$>$' is an abbreviation of 'is bigger than', and the special symbol '$\ngtr$' is an abbreviation of 'is *not* bigger than'. See next section for more details.

[1] The Eiffel Tower tops the list of the "greatest hits in architecture" – http://www.greatbuildings.com/greatest_hits.html.

(2) the use of **variables** – e.g., ‘x’, ‘y’, ‘z’ – together with the understanding that these variables are (implicitly) attached to **universal quantifiers**.

As the intro logic student is aware, variables play a role in logic and mathematics similar to the role played in ordinary language by pronouns and pronoun-like "devices". For example, the sentence

if x is bigger than y, then y is not bigger than x

is understood as short for the more proper

for any x, for any y: if x is bigger than y, then y is not bigger than x

which may be translated into ordinary English as follows.

for any individual x, (and) for any individual y,
if the first individual (i.e., x) is bigger than the second individual (i.e., y),
then the second individual (i.e., y) is *not* bigger than the first individual (i.e., x).

If we abbreviate the theoretical words, and we also abbreviate the logical words, then we can render the above sentence very compactly as follows.

$$\forall x\, \forall y\, \{\, x > y \;\rightarrow\; y \not> x \,\}$$

3. Some Notational Shortcuts

As already noted, in the statement of the laws of size, we adopt a number of notational abbreviations, which we now list more fully.[2]

N1:	$x > y$	=:	x is bigger than y
N2:	$x < y$	=:	x is smaller than y
N3:	$x \geqslant y$	=:	x is bigger than or equal in size to y
N4:	$x \leqslant y$	=:	x is smaller than or equal in size to y
N5:	$x \equiv y$	=:	x is equal in size to y

In addition to these, there are several other abbreviations that we adopt, including the following.

N6:	$x \not> y$	=:	$\sim[x > y]$
N7:	$x \not< y$	=:	$\sim[x < y]$
N8:	$x \neq y$	=:	$\sim[x = y]$
N9:	$\forall x\, \Phi$	=:	for any individual x Φ
N10:	$\exists x\, \Phi$	=:	there is an individual x such that Φ
N11:	$\sim \Phi$	=:	it is not the case that Φ
N12:	$\Phi_1 \,\&\, \Phi_2$	=:	Φ_1 and Φ_2
N13:	$\Phi_1 \vee \Phi_2$	=:	Φ_1 or Φ_2
N14:	$\Phi_1 \rightarrow \Phi_2$	=:	if Φ_1 then Φ_2
N15:	$\Phi_1 \leftrightarrow \Phi_2$	=:	Φ_1 if and only if Φ_2
N14:	$x = y$	=:	x is (identical to) y

[2] It is common practice to introduce special symbols for important theoretical concepts. For example, in arithmetic, one does not insist on writing ‘one’, ‘two’, ‘three’, ‘plus’, ‘times’, but instead writes ‘1’, ‘2’, ‘3’, ‘+’, ‘×’. There is nothing magical in these symbols; they are merely labor-saving devices.

4. A Proposed Size Theory

Now, on the Euclidean paradigm, a theory of size constitutes an attempt on our part to unify and systematize all the laws of size. The theory we propose – GST – is formulated in terms of one primitive concept – “is bigger than”, and two primitive postulates (axioms) given as follows.

Primitive Concept: “is bigger than”

Axiom 1: for any three things, if the first is *not* bigger than the second, and the second is *not* bigger than the third, then the first is *not* bigger than the third

Axiom 2: for any two things: if the first is bigger than the second, then the second is *not* bigger than the first

Using the above notational shortcuts, and using notation from elementary logic, we can formulate the axioms as follows.

A1: $\forall x \forall y \forall z \, \{ \, (x \not> y \;\; \& \;\; y \not> z) \; \rightarrow \; x \not> z \, \}$

A2: $\forall x \forall y \, \{ \, x > y \; \rightarrow \; y \not> x \, \}$

These can be further abbreviated as follows.[3]

A1: $\forall xyz \, \{ \, x \not> y \;\; \& \;\; y \not> z \;\; .\!\!\rightarrow \; x \not> z \, \}$

A2: $\forall xy \, \{ \, x > y \; \rightarrow \; y \not> x \, \}$

5. Some Definitions from Logic (Common Notions)

Before we proceed, it will be useful to introduce some basic terminology from logic, which provide some of the common notions of our theory of size.

C1: A relation R is said to be ***transitive*** exactly if the following obtains:
$\forall xyz \, \{ \, Rxy \;\; \& \;\; Ryz \; .\!\!\rightarrow Rxz \, \}$

C2: A relation R is said to be ***symmetric*** exactly if the following obtains:
$\forall xy \, \{ \, Rxy \; \rightarrow Ryx \, \}$

C3: A relation R is said to be ***asymmetric*** exactly if the following obtains:
$\forall xy \, \{ \, Rxy \; \rightarrow \; \sim Ryx \, \}$

C4: A relation R is said to be ***reflexive*** exactly if the following obtains:
$\forall x \, \{ \, Rxx \, \}$

[3] The multiple-quantifier ‘$\forall xy$’ is short for ‘$\forall x \forall y$’; ‘$\forall xyz$’ is short for ‘$\forall x \forall y \forall z$’; etc. The use of dots adjacent to connectives enables one to reduce the number of parentheses required to punctuate a formula. A dot basically marks (1) the dominant connective, (2) a missing parenthesis, and (3) a logical pause (just like a comma in ordinary language).

C5	A relation R is said to be ***irreflexive*** exactly if the following obtains: $\sim\exists x \{ Rxx \}$
C6:	A relation R is said to be ***connexive*** exactly if the following obtains: $\forall xy \{ Rxy \vee Ryx \}$

These definitions may be abbreviated as follows.

C1:	R is transitive	$=_{df}$	$\forall xyz\{ Rxy \mathbin{\&} Ryz .\!\rightarrow Rxz \}$
C2:	R is symmetric	$=_{df}$	$\forall xy\{ Rxy \rightarrow Ryx \}$
C3:	R is asymmetric	$=_{df}$	$\forall xy\{ Rxy \rightarrow \sim Ryx \}$
C4:	R is reflexive	$=_{df}$	$\forall x \{ Rxx \}$
C5:	R is irreflexive	$=_{df}$	$\sim\exists x \{ Rxx \}$
C6:	R is connexive	$=_{df}$	$\forall xy \{ Rxy \vee Ryx \}$

The special symbol '$=_{df}$' – which is read "equals by definition" – basically means that the expression on the left, called the *definiendum*, is defined to mean the same thing as the expression on the right, called the *definiens*.

There are a number of further concepts that be defined in terms of these notions

C7:	R is an ordering relation	$=_{df}$	R is asymmetric and transitive
C8:	R is a quasi-ordering relation	$=_{df}$	R is reflexive and transitive
C9:	R is a pre-linear-ordering relation	$=_{df}$	R is connexive and transitive
C10:	R is an equivalence relation	$=_{df}$	R is reflexive, symmetric, and transitive

6. Theorems

Closely related to the term 'theo**ry**' is the associated term 'theo**rem**'; Every theory has theorems, which are precisely all those statements that can be logically *logically derived* (*deduced*) from the axioms of the theory, together with the definitions and common notions.

> Let $\mathfrak{T}$ be a theory with axioms $A_1, \ldots, A_k$, and definitions $D_1, \ldots, D_m$, and common notions $C_1, \ldots, C_n$. Then a *theorem* of $\mathfrak{T}$ is, by definition, any statement that can be logically deduced from the set $\{A_1, \ldots, A_k, D_1, \ldots, D_m, C_1, \ldots, C_n\}$

Since the definitions and common notions are often *taken for granted* in many discussions, it is common to simplify the above description by saying that the theorems of $\mathfrak{I}$ are the statements that logically follow from the axioms. This is not strictly speaking correct, but it is convenient shorthand.

Next, notice that the above definition has the following trivial consequence.

> Every axiom of $\mathfrak{I}$ is also a theorem of $\mathfrak{I}$.

Now, the fundamental idea is that, in proposing a theory to systematize a body of knowledge – for example, the laws of size – one strives to deduce all the laws from the axioms, definitions, and common notions. In other words, it is hoped that every law of size is a theorem of the proposed theory of size.

We will not explicitly cite the two axioms as theorems; nor will we cite all the theorems, since there are infinitely-many! Rather, we will cite a few prominent ones. The first two such theorems restate the axioms using the logical notions defined in Section 5. In the language of Euclid, the definitions from Section 5 count as "common notions", and may be freely employed in the development of a theory.[4]

T1: $\not>$ is transitive
T2: $>$ is asymmetric

Although these derivations are "trivial", involving simply the application of the definitions, for the sake of illustration of how one employs axioms, definitions, and common notions in proofs, we offer the following proofs.

(1)	~~SHOW~~: $\not>$ is transitive	DD
(2)	$\forall xyz\,\{x \not> y \;\&\; y \not> z \;.\!\rightarrow\; x \not> z\}$	A1
(3)	$\not>$ is transitive	2,C1

(1)	~~SHOW~~: $>$ is asymmetric	DD
(2)	$\forall xy\,\{x > y \;\rightarrow\; y \not> x\}$	A2
(3)	$\forall xy\,\{x > y \;\rightarrow\; \sim[y > x]\}$	2,N6
(4)	$>$ is asymmetric	3,C3

The Rules of constructing proofs of theorems includes the usual rules from intro logic, plus the following further principles.

> (p1) At any point in a proof, one is entitled to write down any axiom.
>
> (p2) At any point in a proof, one can apply a definition ($=_{\text{df}}$) or shortcut ($=\!:$) to an available line just like a rule of inference.

Notice in both of the above proofs, line (2) applies (p1). The other lines apply (p2), treating C1, C3, and N6 as rules.[5]

[4] In the sense intended, the development of a theory is like the development of a theme or motif in a musical piece, for example, a sonata. One uncovers hidden material implicit in the theme.
[5] Moreover, unlike rules of inference, shortcut rules can be applied to *parts of lines*, which gives them more power.

The next theorem is not quite so straightforward.

T3: $\sim\exists x\{\, x > x \,\}$

Or in ordinary English:

nothing is bigger than itself!

By way of showing that T3 is a theorem of GST, we offer the following proof.

(1)	~~SHOW~~: $\sim\exists x\{\, x>x \,\}$	ID
(2)	$\exists x\{\, x>x \,\}$	As
(3)	~~SHOW~~: ✕	DD
(4)	$a>a$	2,∃O
(5)	$\forall xy\{\, x>y \rightarrow y\ngtr x \,\}$	A2
(6)	$a>a \rightarrow a\ngtr a$	5,∀O×2
(7)	$a\ngtr a$	6,7,→O
(8)	$\sim[a>a]$	7,N6
(9)	✕	4,8,✕I

The following are immediate corollaries[6] of T3.

T4: $\forall x\ \{\, x \ngtr x \,\}$
T5: $\ngtr$ is reflexive
T6: $>$ is irreflexive

From A1 and T5, we have the following as an immediate corollary.

T7: $\ngtr$ is a quasi-ordering relation

We know that $\ngtr$ is transitive (A1), but we don't yet know whether $>$ is transitive. But the following can be shown (exercise!).

T8: $\forall xyz\ \{\, x>y\ \&\ y>z\ .\!\rightarrow\ x>z \,\}$
T9: $>$ is transitive
T10: $>$ is an ordering relation

We can also prove the following theorems.

T11: $\forall xy\ \{\, x\ngtr y\ \text{ or }\ y\ngtr x \,\}$
T12: $\ngtr$ is connexive
T13: $\ngtr$ is a linear-ordering

[6] The word 'corollary' means "a proposition that follows with little or no proof required from one already proven." Usually a corollary is something that follows rather directly; however, a certain vagueness accompanies the word, since directness is a matter of degree.

7. Definitions

In his theory of geometry, Euclid employs (1) common notions, (2) definitions, and (3) postulates (axioms). The proposed theory of size – GST – also employs these three items. In addition, *we* employ notational abbreviations in the presentation of GST.

A key component of an axiomatic reduction involves defining all the relevant concepts in terms of a few primitive concepts plus logic. The theory GST employs exactly one primitive concept – "is bigger than", which is represented by the symbol '>'. Since the laws of size involve a number of other concepts, these concepts must all be logically reduced to '>', which is to say that GST must provide definitions of all the size concepts in terms of '>'. We introduce these definitions gradually. The first one is obvious, but must be officially stated nevertheless.

D1: $x<y \quad =_{\mathrm{df}} \quad y>x$

In other words,

as a matter of definition:
x is smaller than y **if and only if** y is bigger than x

As a result of adding this new concept, we can deduce a few more theorems, which parallel our earlier theorems.

(T9)	$\forall xyz\,\{\, x<y \;\&\; y<z \;.\!\to\; x<z \,\}$	[< is transitive]
(T10)	$\forall xy\,\{\, x<y \to y\nless x \,\}$	[< is asymmetric]
(T11)	$\sim\exists x\,\{\, x<x \,\}$	[< is irreflexive]

By way of illustration, we offer a proof of (T10).

(1)	~~SHOW~~: $\forall xy\,\{\, x<y \to y\nless x \,\}$	UD×2
(2)	~~SHOW~~: $a<b \to b\nless a$	CD
(3)	$a<b$	As
(4)	~~SHOW~~: $b\nless a$	5,N6
(5)	~~SHOW~~: $\sim[b<a]$	ID
(6)	$b<a$	As
(7)	~~SHOW~~: ✕	DD
(8)	$b>a$	3,D1
(9)	$a>b$	6,D1
(10)	$\forall xy\,\{\, x>y \to y\ngtr x \,\}$	A2
(11)	$b>a \to a\ngtr b$	10,∀O×2
(12)	$a\ngtr b$	8,11,→O
(13)	$\sim[a>b]$	12,N6
(14)	✕	9,13,✕I

The next two definitions basically amount to introducing alternative notation, but they turn out to be *very* convenient.

D2: $x \leqslant y \quad =_{\mathrm{df}} \quad x \ngtr y$
D3: $x \geqslant y \quad =_{\mathrm{df}} \quad x \nless y$

D11: The following are some immediate theorems about $\leqslant$ and $\geqslant$.

(T12) $\forall xyz\ \{\ x \leqslant y\ \ \&\ \ y \leqslant z\ \ .\!\rightarrow\ x \leqslant z\ \}$ [$\leqslant$ is transitive]
(T13) $\forall x\ \{\ x \leqslant x\ \}$ [$\leqslant$ is reflexive]
(T14) $\forall xy\ \{\ x \leqslant y\ \vee\ y \leqslant x\ \}$ [$\leqslant$ is connexive]
(T15) $\leqslant$ is a linear ordering

(T16) $\forall xyz\ \{\ x \geqslant y\ \ \&\ \ y \geqslant z\ \ .\!\rightarrow\ x \geqslant z\ \}$ [$\geqslant$ is transitive]
(T17) $\forall x\ \{\ x \geqslant x\ \}$ [$\geqslant$ is reflexive]
(T18) $\forall xy\ \{\ x \geqslant y\ \vee\ y \geqslant x\ \}$ [$\geqslant$ is connexive]
(T19) $\geqslant$ is a pre-linear ordering

The next concept is a little more robust – the concept of "equally big" or "sameness of size". Can one define this concept in terms of the initial concept of "is bigger than"? The answer is a definite affirmative. The following is the official definition.

D4: $x \equiv y \quad =_{\text{df}} \quad x \leqslant y\ \&\ y \leqslant x$

Alternatively stated, x and y are equally big precisely if neither is bigger than the other (check this out yourself). With this definition, we can prove some further theorems, given as follows.

(T20) $\forall x\ \{\ x \equiv x\ \}$ [$\equiv$ is reflexive]
(T21) $\forall xy\ \{\ x \equiv y\ \rightarrow y \equiv x\ \}$ [$\equiv$ is symmetric]
(T22) $\forall xyz\ \{\ x \equiv y\ \ \&\ y \equiv z\ \ .\!\rightarrow\ x \equiv z\ \}$ [$\equiv$ is transitive]

(T23) $\equiv$ is an equivalence relation

The world is full of relations of this kind. Most of them can be formulated using the word 'equally' or 'same'. The following are a few examples.

equally tall, equally heavy, equally wealthy, equally smart, …

same height, same weight, same wealth, same intelligence, …

The reader is invited to supply his/her own examples as well, including ones that don't overtly involve the words 'equally' or 'same'.

We conclude by noting a few remaining theorems of GST, which are laws of size.

(T24) $\forall xy\ \{\ x > y\ \vee\ y > x\ \vee\ x \equiv y\ \}$
(T25) $\forall xyz\ \{\ x > y\ \ \&\ \ y \equiv z\ \ .\!\rightarrow\ x > z\ \}$
(T26) $\forall xyz\ \{\ x \equiv y\ \ \&\ \ y > z\ \ .\!\rightarrow\ x > z\ \}$