Intermediate Logic
Rules of Derivation

1. Sentential Logic

Henceforth, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are closed formulas.

1. Inference Rules

&I		**&O**		~**&**I/O
$\mathcal{A}$ $\mathcal{B}$ ——— $\mathcal{A}\&\mathcal{B}$		$\mathcal{A}\&\mathcal{B}$ ——— $\mathcal{A}$	$\mathcal{A}\&\mathcal{B}$ ——— $\mathcal{B}$	$\sim(\mathcal{A}\&\mathcal{B})$ ═════ $\mathcal{A} \rightarrow \sim\mathcal{B}$

vI		**vO**		~**v**I/O
$\mathcal{A}$ ——— $\mathcal{A}\vee\mathcal{B}$	$\mathcal{B}$ ——— $\mathcal{A}\vee\mathcal{B}$	$\mathcal{A}\vee\mathcal{B}$ $\sim\mathcal{A}$ ——— $\mathcal{B}$	$\mathcal{A}\vee\mathcal{B}$ $\sim\mathcal{B}$ ——— $\mathcal{A}$	$\sim(\mathcal{A}\vee\mathcal{B})$ ═════ $\sim\mathcal{A}\ \&\ \sim\mathcal{B}$

→I		→O		~→I/O
$\sim\mathcal{A}$ ——— $\mathcal{A}\rightarrow\mathcal{B}$	$\mathcal{B}$ ——— $\mathcal{A}\rightarrow\mathcal{B}$	$\mathcal{A}\rightarrow\mathcal{B}$ $\mathcal{A}$ ——— $\mathcal{B}$	$\mathcal{A}\rightarrow\mathcal{B}$ $\sim\mathcal{B}$ ——— $\sim\mathcal{A}$	$\sim(\mathcal{A}\rightarrow\mathcal{B})$ ═════ $\mathcal{A}\ \&\ \sim\mathcal{B}$

« I		« O		~ « I/O
$\mathcal{A}\rightarrow\mathcal{B}$ $\mathcal{B}\rightarrow\mathcal{A}$ ——— $\mathcal{A}\leftrightarrow\mathcal{B}$		$\mathcal{A}\leftrightarrow\mathcal{B}$ ——— $\mathcal{A}\rightarrow\mathcal{B}$	$\mathcal{A}\leftrightarrow\mathcal{B}$ ——— $\mathcal{B}\rightarrow\mathcal{A}$	$\sim(\mathcal{A}\leftrightarrow\mathcal{B})$ ═════ $\sim\mathcal{A}\leftrightarrow\mathcal{B}$

※I	※O	**DN**
$\mathcal{A}$ $\sim\mathcal{A}$ ——— ※	※ ——— $\mathcal{A}$	$\sim\sim\mathcal{A}$ ===== $\mathcal{A}$

Note: The ~O and ~I rules are combined, using a long equals sign '==='. Henceforth, any rule that is displayed with '===' is a bi-directional rule, which can be used both as an in-rule and as an out-rule.

2. Strategic Rules

Direct Derivation (DD)

~~SHOW~~: $\mathcal{A}$ DD
$\mathcal{A}$

Indirect Derivation (ID)

~~SHOW~~: $\mathcal{A}$ ID
$\sim\mathcal{A}$ As
~~SHOW~~: ※

Conditional Derivation (CD)

~~SHOW~~: $\mathcal{A}\rightarrow\mathcal{B}$ CD
$\mathcal{A}$ As
~~SHOW~~: $\mathcal{B}$

Tilde Indirect Derivation (~D)

~~SHOW~~: $\sim\mathcal{A}$ ~D
$\mathcal{A}$ As
~~SHOW~~: ※

Ampersand Derivation (&D)

~~SHOW~~: $\mathcal{A}\&\mathcal{B}$ &D
~~SHOW~~: $\mathcal{A}$
~~SHOW~~: $\mathcal{B}$

Biconditional Derivation (« D)

~~SHOW~~: $\mathcal{A}\leftrightarrow\mathcal{B}$ ↔D
~~SHOW~~: $\mathcal{A}\rightarrow\mathcal{B}$
~~SHOW~~: $\mathcal{B}\rightarrow\mathcal{A}$

Wedge Indirect Derivation (ÚID)

~~SHOW~~: $\mathcal{D}_1 \vee \mathcal{D}_2 \vee ... \vee \mathcal{D}_k$ ∨ID
$\sim\mathcal{D}_1$ As
$\sim\mathcal{D}_2$ As
…
$\sim\mathcal{D}_k$
~~SHOW~~: ※

Separation of Cases (SC)

$\mathcal{D}_1 \vee \mathcal{D}_2 \vee ... \vee \mathcal{D}_k$
~~SHOW~~: C SC
c1: $\mathcal{D}_1$ As
~~SHOW~~: C
c2: $\mathcal{D}_2$ As
~~SHOW~~: C
.
.
.
ck: $\mathcal{D}_k$ As
~~SHOW~~: C

2. Classical Quantifier Logic

1. Basics

In what follows, Φ is a formula, in which ν is the only variable (if any) that occurs free, and $\Phi[\varepsilon/\nu]$ is the formula that results when ε replaces every occurrence of ν that is free in Φ. An expression is ***closed*** iff it contains no *free occurrence* of any variable. An occurrence of a variable ν is ***free*** in expression $\mathcal{E}$ iff that occurrence does not lie within the scope of an operator binding ν – i.e., $\forall\nu$, $\exists\nu$, or $\iota\nu$.[1]

2. Quantifier Rules

Universal-Out (∀O)	Existential-In (∃I)
$\forall\nu\Phi$ ——— $\Phi[\tau/\nu]$	$\Phi[\tau/\nu]$ ——— $\exists\nu\Phi$
τ is any closed singular-term.	

Universal-Derivation (UD)	Existential-Out (∃O)
~~SHOW~~: $\forall\nu\Phi$ │ ~~SHOW~~: $\Phi[n/\nu]$ │ │	$\exists\nu\Phi$ ——— $\Phi[n/\nu]$
n is any ***new*** constant.	

Quantifier-Negation	
$\sim\forall\nu\Phi$ ═════ $\exists\nu\sim\Phi$	$\sim\exists\nu\Phi$ ═════ $\forall\nu\sim\Phi$

Tilde-Universal-Out (~∀O) ~∀O = QN+∃O	**Tilde-Existential-Out** (~∃O) ~∃O = QN+∀O
$\sim\forall\nu\Phi$ ——— $\sim\Phi[n/\nu]$ n is any ***new*** constant.	$\sim\exists\nu\Phi$ ——— $\sim\Phi[\tau/\nu]$ τ is any closed singular-term.

A constant counts as ***old*** precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as ***new***.

[1] The new symbol 'ι' (upside-down iota) is the **definite description operator**, which arises later. For *current* purposes, a singular-term counts as closed precisely when it contains no variable.

3. Classical Identity Logic

1. Simple Identity Rules – Identity is an Equivalence Relation

Reflexivity (R=)	**Symmetry** (S=)	**Transitivity** (T=)
$\varnothing$ ——— $\sigma = \sigma$	$\sigma = \tau$ ——— $\tau = \sigma$	$\rho = \sigma$ $\sigma = \tau$ ——— $\rho = \tau$
σ, τ, and ρ are any closed singular-terms.		

2. Leibniz's Law

LL	
$\sigma = \tau$ $\Phi[\sigma/\nu]$ ——— $\Phi[\tau/\nu]$	$\sigma = \tau$ $\Phi[\tau/\nu]$ ——— $\Phi[\sigma/\nu]$

4. Free Description Logic

1. Introduction

Classical first-order logic is based on the following two presuppositions:

(C1) The domain (universe) of discourse is not empty; accordingly, the sentence ‘there is something’ is logically true, even though it is not necessarily true.

(C2) Every singular term, no matter how silly, denotes an existing object (i.e., element of the domain).

In contrast to classical logic, there is ***Free Logic***, of which there are two variants. The more radical version (Universally-Free Logic) denies both (C1) and (C2). The less radical version of Free Logic denies (C2), but accepts (C1). In what follows, we pursue the more radical variant.

2. Constants versus Proper Nouns

In intro logic, the distinction between unquantified variables ("constants") (‘*a*’, ‘*b*’, ‘*c*’, etc.) and proper nouns (‘Jay’, ‘Kay’, ‘the U.S.’, etc.) is *not* important. By contrast, in free logic, the distinction is **very important**. In particular, whereas constants always denote existing objects (in the domain of quantification), proper nouns need not denote anything. In doing derivations in free logic, one treats constants as purely intra-derivational symbols. In particular, we have the following definition.

A ***constant*** is an atomic singular-term that is introduced by UD or ∃O.
Any atomic singular-term that occurs in the premises or conclusion is regarded as a **proper noun**, not a constant.
A constant counts as ***old*** precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as ***new***. (as before)

3. Quantifier Rules

Universal-Out (∀O)	**Existential-In** (∃I)
$\forall\nu\Phi$ ——— $\Phi[o/\nu]$	$\Phi[o/\nu]$ ——— $\exists\nu\Phi$
o is any ***old*** constant.	

Universal-Derivation (UD)	**Existential-Out** (∃O)
~~SHOW~~: $\forall\nu\Phi$ | ~~SHOW~~: $\Phi[n/\nu]$ | |	$\exists\nu\Phi$ ——— $\Phi[n/\nu]$
n is any ***new*** constant.	

Quantifier-Negation (QN)	
$\sim\forall\nu\Phi$ ═══ $\exists\nu\sim\Phi$	$\sim\exists\nu\Phi$ ═══ $\forall\nu\sim\Phi$

Tilde-Universal-Out (~∀O) ~∀O = QN+∃O	**Tilde-Existential-Out** (~∃O) ~∃O = QN+∀O
$\sim\forall\nu\Phi$ ——— $\sim\Phi[n/\nu]$	$\sim\exists\nu\Phi$ ——— $\sim\Phi[o/\nu]$
n is any ***new*** constant.	*o* is any ***old*** constant.

4. Description Rules

Iota-Out (ιO)	**Iota-In** (ιI)
$c = \iota\nu\Phi$ ——— $\forall\nu(\Phi \leftrightarrow \nu{=}c)$	$\forall\nu(\Phi \leftrightarrow \nu{=}c)$ ——— $c = \iota\nu\Phi$
c must be a ***constant***.	

5. Short-Cut Rules

1. Short-Cut Rules for Quantifier Logic

1. The Rule SL

If a line can be derived from available lines using only SL rules, then it may be written down by the rule SL. This rule may be used in place of any combination of SL rules, including inference rules and show rules.

2. The Immediate Show-Cancel Rule

If a show-line follows from available lines (**earlier** or **later**!) by a rule, then it can be cancelled by that rule. Annotation: cite the line number(s) and the rule.

3. The Conjunction Rule

Any available conjunctive line (with any number of conjuncts) can be treated as the appropriate number of separate lines, numbered (e.g.) 7a, 7b, 7c. And conversely, any number of available lines can be treated as the corresponding conjunction.

4. Rule-Multiplication

Any one-place rule can be multiplied, **provided** the particular rule also applies to the intermediate line. For example, ∀O+∀O = ∀O2; UD+UD = U2D; ~∃O+~∃O = ~∃O2.

5. Further Rule Combinations

∀O, ∀O2, etc. can be combined with →O to produce ∀→O, ∀2→O, etc.,
and UD, UD2, etc., can be combined with CD to produce UCD, U2CD, etc.
Similarly, UD, UD2, etc., can be combined with ↔D to produce UBD, U2BD, etc.

6. Contraposition Rule

For every *genuine* one-place rule ℝ, there is an associated contrapositive rule ℝ(–), which is obtained by reversing and negating and the premise and conclusion.

NOTE CAREFULLY:
∃O is *not* a genuine *inference rule*, but is rather an ***assumption rule.***

2. Short-Cut Rules for Identity Logic

1. The Rule QL

If a line can be derived from previous available lines using only quantifier rules, but it cannot be derived using just SL rules, then it may be written down by the rule QL. This rule may be used in place of any QL inference rule, as well as any combination of QL rules, including inference rules and show rules.
NOTE CAREFULLY: The rule ∃O, and hence ~∀O, are **not** genuine inference rules, but **assumption rules**. No instance of ∃O is valid by QL! So when you cite ∃O or ~∀O, **do not** cite it as QL.

2. Rule EQ(=)

If a line can be derived from previous available lines using only the simple identity-rules [i.e., Ref=, Sym=, Tran=] together with SL, then it may be written down by the rule EQ(=) [short for '= is an equivalence relation'].

3. Short-Cut Rules for Description Logic

LL(–)

$\Phi[\sigma/\nu]$ $\sim\Phi[\tau/\nu]$ ――――― $\sigma \neq \tau$	$\sim\Phi[\sigma/\nu]$ $\Phi[\tau/\nu]$ ――――― $\sigma \neq \tau$

Only-Out (OO)

$\forall\nu(\Phi \leftrightarrow \nu{=}c)$
―――――
$\Phi[c/\nu]$

c is a constant.

Alphabetic Variance (AV)

$\Phi[\boldsymbol{u}]$
―――
$\Phi[\boldsymbol{v}]$

Here, ***u***, ***v*** are variables, Φ[***u***] is a formula in which ***v*** does *not* occur, Φ[***v***] is a formula in which ***u*** does *not* occur, and Φ[***v***] results when every occurrence of ***u*** in Φ[***u***] is replaced by an occurrence of ***v***.