Quantum Electromagnetic Field

Axiomatically, the quantum electromagnetic field is introduced as the quantum field of bosons (called photons), with the (non-interacting) Hamiltonian

$$H = \sum_{k} \sum_{\lambda=1,2} \hbar \omega_k (\hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda} + \hat{a}_{k\lambda} \hat{a}_{k\lambda}^\dagger)/2, \quad \omega_k = ck.$$  \hspace{1cm} (1)

Each single-photon mode is characterized by the wave vector $k$ (same as momentum, up to the factor $\hbar$) and a polarization. For each momentum, the polarization subspace is two-dimensional.\(^1\) In accordance with (1), each single-photon mode is a quantum harmonic oscillator of the frequency $\omega_k = ck$ (with $c$ the velocity of light). Note that the two-fold degeneracy of the energy with respect to polarization implies a freedom of choosing the polarization basis.

The next postulate concerns the interaction of the quantum electromagnetic field with charges. The crucial quantity here is the notion of the operator of the vector potential defined as follows:

$$\hat{A}(r) = \sqrt{4\pi c} \sum_{k} \sum_{\lambda=1,2} \sqrt{\frac{\hbar}{2 \omega_k}} \left( \hat{a}_{k\lambda} \hat{e}_{k\lambda} e^{ikr} + \hat{a}_{k\lambda}^\dagger \hat{e}_{k\lambda}^* e^{-ikr} \right),$$  \hspace{1cm} (2)

where $\hat{e}_{k\lambda}$'s are the unit ($||\hat{e}_{k\lambda}|| = 1$) polarization vectors satisfying the conditions of orthogonality:

$$\hat{e}_{k,1} \cdot \hat{e}_{k,2} = 0$$  \hspace{1cm} (3)

and transversality:

$$\hat{e}_{k\lambda} \cdot k = 0.$$  \hspace{1cm} (4)

In view of the degeneracy of the photon energy in terms of the polarization, the vectors $\hat{e}_{k\lambda}$ can be selected to be both real and complex. The square-root prefactor in (2) is a matter of units convention. The transversality condition for the polarization vectors, Eq. (4), implies the transversality condition (aka Coulomb gauge) for the vector potential:

$$\nabla \cdot \hat{A} = 0.$$  \hspace{1cm} (5)

\(^1\)The precise physical meaning of the notion of polarization will become clear later, when we will introduce the field operator of vector potential in the context of coupling of electromagnetic field to charged matter.
The coupling to a charged particle/charged quantum field is postulated to be described by extending the single-particle momentum operator, \( \hat{p} = -i\hbar \nabla \), by the following rule (similar to its classical counterpart):

\[
\hat{p} \rightarrow \hat{p} + (e/c)\hat{A}(r).
\]  

(6)

To be specific, let us talk of electrons in which case \( e \) is the electron charge. In terms of the energy, we have\(^2\) (for briefness, we omit the dependence of \( \hat{A} \) on \( r \))

\[
\frac{\hat{p}^2}{2m} \rightarrow \frac{1}{2m} \left( \hat{p} + \frac{e}{c} \hat{A} \right)^2 = \frac{\hat{p}^2}{2m} - \frac{i\hbar e}{mc} \hat{A} \cdot \nabla + \frac{e^2}{2mc^2} \hat{A}^2.
\]  

(7)

The first term in the r.h.s. of (7) is the operator of kinetic energy of the electron. The other two terms describe the interaction with the electromagnetic field:

\[
H_{\text{int}} = -\frac{i\hbar e}{mc} \hat{A} \cdot \nabla + \frac{e^2}{2mc^2} \hat{A}^2.
\]  

(8)

By the standard rules of the second quantization, the interaction with a single electron is readily promoted to the interaction with the quantum field of electrons:

\[
\hat{\Psi}^\dagger (-i\hbar \nabla)^2 \hat{\Psi} \rightarrow \hat{\Psi}^\dagger [ -i\hbar \nabla + (e/c)\hat{A} ]^2 \hat{\Psi},
\]

and

\[
H_{\text{int}} = -\frac{i\hbar e}{mc} \int \hat{\Psi}^\dagger \hat{A} \cdot \nabla \hat{\Psi} \, d^3r + \frac{e^2}{2mc^2} \int \hat{A}^2 \hat{\Psi}^\dagger \hat{\Psi} \, d^3r.
\]  

(9)

In the Heisenberg picture,

\[
\hat{a}_{k\lambda}(t) = \hat{a}_{k\lambda} e^{-i\omega_k t}
\]

and, thus,

\[
\hat{A}(r, t) = \sqrt{4\pi c} \sum_k \sum_{\lambda=1,2} \sqrt{\frac{\hbar}{2\omega_k}} \left( \hat{a}_{k\lambda} \hat{e}_{k\lambda} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega_k t} + \hat{a}_{k\lambda}^\dagger \hat{e}_{k\lambda}^* e^{-i\mathbf{k} \cdot \mathbf{r} + i\omega_k t} \right).
\]  

(10)

For certain applications, it is convenient to introduce field operators of electric and magnetic fields. The magnetic field is defined by

\[
\hat{B}(r) = \nabla \times \hat{A}.
\]  

(11)

\(^2\)The term with \( \nabla \cdot \hat{A} \) is absent in view of Eq. (4).
In the Heisenberg picture, the electric field is conveniently defined as
\[ \hat{E}(r, t) = -\frac{1}{c} \frac{\partial \hat{A}(r, t)}{\partial t}. \] (12)

In the Schrödinger picture we have
\[ \hat{E}(r) = i\sqrt{\frac{2}{4\pi}} \sum_k \sum_{\lambda=1,2} \sqrt{\frac{\hbar \omega_k}{2}} \left( \hat{a}_{k\lambda} \hat{e}_{k\lambda} e^{ikr} - \hat{a}_{k\lambda}^\dagger \hat{e}_{k\lambda}^* e^{-ikr} \right). \] (13)

Equations (12) and (13) are equivalent to each other. Any of the two can be used as a definition of \( \hat{E} \) implying its counterpart as a theorem.

In terms of the electric and magnetic fields, the Hamiltonian (1) is expressed by the formula
\[ H = \frac{1}{8\pi} \int (\hat{E}^2 + \hat{B}^2) d^3 r, \]
which is a direct analogy of the expression for the energy of the classical electromagnetic field.

As is readily checked, the fields \( \hat{A}(r, t), \hat{E}(r, t), \) and \( \hat{B}(r, t) \) satisfy Maxwell equations. In particular, all of them satisfy the relativistic wave equation. This is not surprising in view of the combination of “standard” definitions (11)–(12) with the fact the Hamiltonian (1) and Eq. (2) feature the correspondence between (bosonic) creation/annihilation operators and classical complex canonical variables:
\[ \sqrt{\hbar} \hat{a}_{k\lambda} \leftrightarrow a_{k\lambda} \quad \sqrt{\hbar} \hat{a}_{k\lambda}^\dagger \leftrightarrow a_{k\lambda}^*. \] (14)

In terms of complex canonical variables, the Hamiltonian of free classical electromagnetic field reads
\[ H = \sum_k \sum_{\lambda=1,2} \omega_k |a_{k\lambda}|^2 \quad \text{(classical)}, \]
and from the equations of motion we find
\[ i\dot{a}_{k\lambda} = \frac{\partial H}{\partial a_{k\lambda}^*} \quad \Rightarrow \quad a_{k\lambda}(t) = a_{k\lambda}(0)e^{-i\omega_k t}, \]
which yields the standard solution to the Maxwell equations for the free classical electromagnetic field
\[ \hat{A}(r, t) = \sqrt{\frac{4\pi}{c}} \sum_k \sum_{\lambda=1,2} \frac{1}{2\omega_k} \left[ a_{k\lambda}(0) \hat{e}_{k\lambda} e^{ikr - i\omega_k t} + a_{k\lambda}^*(0) \hat{e}_{k\lambda}^* e^{-ikr + i\omega_k t} \right]. \] (15)
The quantum-to-classical correspondence (14) naturally leads to the classical-field behavior of the quantum field at large occupation numbers.

**Emission rate of an excited atom**

Let us use the interaction (8) to calculate the emission rate of an atom in an excited electronic state. We confine ourselves to a hydrogen-atom-type problem assuming that there is only one electron in the system. Correspondingly, there are only two electronic states involved: the initial (excited) state \( \varphi_i(r) \) and the final (ground) state \( \varphi_f(r) \). The calculation can be done by the Golden Rule. The applicability of the perturbation theory here is guaranteed by two small parameters, one of which is the fine-structure constant

\[
\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \ll 1.
\]  

In quantum electrodynamics, the fine-structure constant plays the role of the coupling constant between the electromagnetic field and charged matter. The smallness of \( \alpha \) allows one to controllably use perturbative methods. In non-relativistic problems like ours, there is yet another small parameter,

\[
\frac{v}{c} \ll 1,
\]

where \( v \) is the characteristic velocity of charged particle(s). The denominators of the first and the second terms in (8) imply that the first term involves the parameter (17) to the first power, while the second term is proportional to the square of the non-relativistic parameter. Hence, the second term is small and we can safely omit it. We are left with the interaction of the form

\[
V = -i\frac{\hbar e}{mc} \hat{A} \cdot \nabla = -\frac{e}{c} \hat{A} \cdot \hat{v},
\]

with \( \hat{v} = \hat{p}/m \) the operator of velocity of the electron. An extra approximation—also controlled by the non-relativistic parameter (17)—is the replacement

\[
\hat{A}(r) \to \hat{A}(r = 0).
\]

Condition (17) guaranties that the momentum of the emitted photon is much smaller than the typical momentum of the electron. By the uncertainty
principle, the latter is on the order of the size of the electronic wave function, \( R_0 \). Hence, for the relevant exponentials in the operator \( \hat{A}(r) \) we have
\[
e^{i k r} \approx 1 \quad \text{at} \quad r \sim R_0.
\]
Now recall that the kinematic part of this calculation (i.e., everything dealing with the conservation of energy and momentum, but not with the particular form of the matrix element) has been performed already in Problem 19. We thus take the result of Problem 19 and substitute the matrix element corresponding to the interaction (18). We should be careful though with a simplifying assumption made in Problem 19. Namely, the assumption that the emission of a photon was isotropic. In fact, this is not the case. The excited state of the atom is a \( p \)-state (the state with the momentum \( l = 1 \)) implying that the matrix element between the initial and final states involves a vector:
\[
\mathbf{v}_{fi} = \langle \varphi_f | \hat{\mathbf{v}} | \varphi_i \rangle.
\]  
Hence, the differential rate of the emission of the photon should be sensitive to the direction of \( \mathbf{v}_{fi} \) (being polarization-dependent for the very same reason), so that it makes sense to discuss the differential rather than total rate. For the differential emission rate for a given polarization \( \lambda \), the result of the Problem 19 is
\[
dW_\lambda = |g|^2 \frac{\omega^2 \hbar^2}{4 \pi^2 \hbar^2 c^3} d\Omega_k.
\]
Here we replaced \( E_\ast \) with \( \hbar \omega \), where \( \omega = ck \) is the photon frequency. Now we make the replacement
\[
g \to \langle f | V | i \rangle = \left( e/c \right) \langle n_{k,\lambda} = 1 | \hat{A}(r = 0) | \text{vac} \rangle : \langle \varphi_f | \hat{\mathbf{v}} | \varphi_i \rangle.
\]
Here \( | \text{vac} \rangle \) is the vacuum state of photons and \( | n_{k,\lambda} = 1 \rangle \) is the state with one photon (in the mode \( k, \lambda \)). With (2) we find
\[
\langle n_{k,\lambda} = 1 | \hat{A}(r = 0) | \text{vac} \rangle = e \sqrt{\frac{2 \pi \hbar}{\omega}} \mathbf{e}_{k\lambda}^*.
\]
Hence,
\[
g \to \langle f | V | i \rangle = e \sqrt{\frac{2 \pi \hbar}{\omega}} \mathbf{e}_{k\lambda}^* \cdot \mathbf{v}_{fi},
\]
and we arrive at the result
\[
dW_\lambda = \frac{\alpha}{2 \pi} \frac{\omega}{\hbar^2 c^2} |\mathbf{e}_{k\lambda}^* \cdot \mathbf{v}_{fi}|^2 d\Omega_k.
\]  
5
A simple dimensional estimates yields
\[ \frac{W}{\omega} \sim \alpha \left( \frac{v}{c} \right)^2 \ll 1, \]
confirming the applicability of the Golden Rule treatment. We conclude that
the decay width \( \Gamma \sim \hbar W \) of the excited state of the atom is much smaller
than the excitation energy \( E_* = \hbar \omega \), so that corresponding Breit-Wigner
resonance (associated with resonant fluorescence) is very sharp.

Speaking of practical calculation of the matrix element \( v_{fi} \) between the
two electronic states, there is a convenient trick of relating this matrix ele-
ment to the matrix element for the electron position vector,
\[
\mathbf{r}_{fi} = \langle \varphi_f | \hat{\mathbf{r}} | \varphi_i \rangle = \int \varphi_f^*(\mathbf{r}) \mathbf{r} \varphi_i(\mathbf{r}) \, d^3r.
\]
The trick utilizes generic relation
\[
\hat{\mathbf{v}} = \frac{1}{i\hbar} [\hat{\mathbf{r}}, H_0],
\]
where \( H_0 \) is the Hamiltonian of the electron with both kinetic and potential
terms, but without coupling to the electromagnetic field. Here we find
\[
v_{fi} = \frac{1}{i\hbar} \langle \varphi_f | [\hat{\mathbf{r}}, H_0] | \varphi_i \rangle = \frac{1}{i\hbar} (E_i^{(0)} - E_f^{(0)}) \langle \varphi_f | \hat{\mathbf{r}} | \varphi_i \rangle = \frac{E_*}{i\hbar} \mathbf{r}_{fi} = -i\omega \mathbf{r}_{fi}.
\]
Equation (20) becomes
\[
dW_\lambda = \frac{\alpha \, \omega^3}{2\pi \, c^2} |\hat{e}_{k\lambda} \cdot \mathbf{r}_{fi}|^2 \, d\Omega_k. \tag{21}
\]