

Complex Variables in Classical Hamiltonian Mechanics

In the classical Hamiltonian formalism, the state of a mechanical system is given by a set of pairs of conjugated variables, of variables $\{q_j, p_j\}$, ($j = 1, 2, \dots$) referred to as coordinates and momenta, respectively. The equations of motion are generated with the Hamiltonian function, $H(\{q_j, p_j\})$, by the following axiom

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad (1)$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}. \quad (2)$$

Some times it proves reasonable to introduce complex variables $\{a_j\}$ related to $\{q_j, p_j\}$ by

$$a_j = (\alpha q_j + i\beta p_j), \quad (3)$$

where α and β are some complex numbers (which can also depend on j , but we are not interested here in such a generalization). The equations of motion then acquire the form

$$i\lambda\dot{a}_j = \frac{\partial H}{\partial a_j^*}, \quad (4)$$

with $\lambda = 1/(\alpha\beta^* + \alpha^*\beta)$. That is each pair of real equations (1)-(2) is replaced with one complex equation (4). One may equivalently use the complex-conjugated equation $-i\lambda\dot{a}_j^* = \partial H/\partial a_j$. Here we take into account that H is real.

Poisson bracket, $\{A, B\}$, is a function of coordinates and momenta constructed from the two other functions, $A(\{q_j, p_j\})$ and $B(\{q_j, p_j\})$ by the following prescription:

$$\{A, B\} = \sum_j \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} - \frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j}. \quad (5)$$

As is seen from (1)-(2)—and this is the most important property of the Poisson brackets,—for any function of coordinate and momenta, $A(\{q_j, p_j\})$, its time derivative during the evolution can be expressed as

$$\dot{A} = \{H, A\}. \quad (6)$$

In particular, if $\{H, A\} \equiv 0$, then the quantity A is a constant of motion.

In complex variables, the definition (5) acquires the form

$$\{A, B\} = (i/\lambda) \sum_j \frac{\partial A}{\partial a_j} \frac{\partial B}{\partial a_j^*} - \frac{\partial A}{\partial a_j^*} \frac{\partial B}{\partial a_j}. \quad (7)$$

Problem A1. Check this. Make sure that $\{a_j, a_k\} = \{a_j^*, a_k^*\} = 0$, $\{a_j, a_k^*\} = (i/\lambda)\delta_{j,k}$.

Problem A2. (a) Derive and solve complex equation of motion for the system $H = \omega|a|^2$ (harmonic oscillator). (Set $\lambda = 1$.—It's just a matter of units in which one measures time.)

(b) Derive complex equations of motion for the system of two (non-trivially) coupled oscillators: $H = \omega_1|a_1|^2 + \omega_2|a_2|^2 + \gamma[(a_1^*)^2 a_2^2 + (a_2^*)^2 a_1^2]$. Do not solve it, but use it to show that the system features the constant of motion $N = |a_1|^2 + |a_2|^2$; show the same with Poisson bracket technique.

The constant of motion N in the Problem A2(b) is actually related to some special symmetry of corresponding Hamiltonian. This symmetry—known as *global $U(1)$ symmetry*—implies invariance of the Hamiltonian with respect to global $U(1)$ transformation, which is simultaneously shifting the phases of all complex variables a_j by one and the same angle θ_0 : $a_j \rightarrow e^{i\theta_0} a_j$. If the global $U(1)$ symmetry takes place, then the following quadratic form of the complex variables

$$N = \sum_j |a_j|^2 \quad (8)$$

is a constant of motion: $\{H, N\} = 0$.

Problem A3. Prove this statement. *Hint.* Start with noticing that in the limit of $\theta_0 \rightarrow 0$, the global $U(1)$ transformation reduces to $H \rightarrow H + \theta_0 \sum_j \partial H / \partial \theta_j$, so that global $U(1)$ invariance implies $\sum_j \partial H / \partial \theta_j \equiv 0$. Don't forget about the relation:

$$\partial / \partial \theta = x \partial / \partial y - y \partial / \partial x = i(z \partial / \partial z - z^* \partial / \partial z^*), \quad (9)$$

where $z = x + iy = |z|e^{i\theta}$.

Note. Complex variables $\{a_j\}$ prove very convenient when finding a quantum counterpart of a classical system. In this procedure, known as *canonical quantization*, the variables a_j and a_j^* are simply replaced with annihilation and creation operators, respectively. The parameter λ now cannot be *arbitrarily* chosen. It is one and the same for *all* quantum systems, and is nothing else than the Planks's constant, \hbar . Upon the quantization procedure,

former Poisson brackets turn out to correspond to commutators between corresponding operators: $\{A, B\} \rightarrow -(i/\hbar)[A, B]$. The quantity N in the quantum system is nothing else than the total number of quanta (particles); it is conserved by $U(1)$ -symmetric Hamiltonians.

Canonical Transformation

Suppose we have a set of complex canonical variables, $\{a_j\}$, and would like to consider another set of variables, $\{b_s\}$, $b_s \equiv b_s(\{a_j\})$. How do the equations of motion look like, if expressed in terms of new variables? The answer is given by the general relation

$$i\dot{b}_s = i\{H, b_s\} = \sum_j \left[\frac{\partial b_s}{\partial a_j} \frac{\partial H}{\partial a_j^*} - \frac{\partial b_s}{\partial a_j^*} \frac{\partial H}{\partial a_j} \right]. \quad (10)$$

The next question is whether it is possible to select new variables in such a way—independently of the particular form of H —that they would be also canonical, that is (10) would be actually equivalent to

$$i\dot{b}_s = \frac{\partial H}{\partial b_s^*}, \quad (11)$$

for *any* H . The answer is positive. To arrive at the necessary and sufficient conditions for the new variables to be canonical we note that basically we require that

$$\sum_j \left[\frac{\partial b_s}{\partial a_j} \frac{\partial}{\partial a_j^*} - \frac{\partial b_s}{\partial a_j^*} \frac{\partial}{\partial a_j} \right] \equiv \frac{\partial}{\partial b_s^*} \equiv \sum_j \left[\frac{\partial a_j^*}{\partial b_s^*} \frac{\partial}{\partial a_j^*} + \frac{\partial a_j}{\partial b_s^*} \frac{\partial}{\partial a_j} \right]. \quad (12)$$

Since the differential operators $\{\partial/\partial a_j, \partial/\partial a_j^*\}$ are linear independent, Eq. (12) immediately leads to

$$\frac{\partial b_s}{\partial a_j} = \frac{\partial a_j^*}{\partial b_s^*}, \quad \frac{\partial b_s}{\partial a_j^*} = -\frac{\partial a_j}{\partial b_s^*}. \quad (13)$$

These are the necessary and sufficient conditions for $\{b_s\}$ to be canonical.

Problem A4. Show that the following two elementary transformations dealing with only one variable, a_j , are canonical.

(a) Shift of the variable: $a_j \rightarrow a_j + c_j$, where c_j is a complex constant.

- (b) Shift of the variable's phase: $a_j \rightarrow e^{i\varphi_j} a_j$, where φ_j is a real constant.
(c) Show that the linear transformation $b_s = \sum_j u_{sj} a_j$ is canonical, if and only if the matrix u is unitary.

A remarkable fact readily following from (13) is that the Poisson brackets are invariant with respect to the canonical transformation. That is the result for $\{A, B\}$ is *the same* for *any* set of canonical variables.

Problem A5. Prove this.

Actually, the invariance of the Poisson brackets under some transformation of variables is a *sufficient* condition for that transformation to be canonical. Indeed, noticing that for any quantity A we have

$$\frac{\partial A}{\partial a_j^*} = i\{A, a_j\}, \quad \frac{\partial A}{\partial a_j} = i\{a_j^*, A\}, \quad (14)$$

and requiring that the Poisson brackets be invariant under our transformation of variables, we immediately arrive at (13):

$$\frac{\partial b_s}{\partial a_j} = i\{a_j^*, b_s\} = \frac{\partial a_j^*}{\partial b_s^*}, \quad (15)$$

$$\frac{\partial b_s}{\partial a_j^*} = i\{b_s, a_j\} = -i\{a_j, b_s\} = -\frac{\partial a_j}{\partial b_s^*}. \quad (16)$$

Bilinear Hamiltonian. Bogoliubov Transformation

Given a Hamiltonian function, one can bilinearize it in the vicinity of its (local) minimum to study the normal modes. The general form of the Hamiltonian after bilinearization is (each variable is now reckoned from its equilibrium value—a shift of a variable is a canonical transformation):

$$H = \sum_{ij} \left(A_{ij} a_i^* a_j + \frac{1}{2} B_{ij} a_i a_j + \frac{1}{2} B_{ij}^* a_i^* a_j^* \right), \quad (17)$$

where without loss of generality we can assume that (the requirement for the Hamiltonian to be real and symmetrization)

$$A_{ij} = A_{ji}^*, \quad B_{ij} = B_{ji}. \quad (18)$$

If a Hamiltonian of the form (17) features a (local) minimum at the point $a_1 = a_2 = a_3 = \dots = 0$, then it can be diagonalized by linear canonical

transformation (Bogoliubov transformation),

$$b_s = \sum_j (u_{sj} a_j + v_{sj} a_j^*) . \quad (19)$$

By “diagonalized” we mean that in terms of the new canonical variables $\{b_s\}$ the Hamiltonian will read

$$H = \sum_s E_s b_s^* b_s , \quad (20)$$

that is will be equivalent to a set of non-interacting harmonic oscillators (normal modes).

Before we show how to relate the u and v coefficients to the matrices A and B , let us first establish their general properties. From the requirement that the transformation (19) be a canonical we get [differentiate both sides with respect to a_j , a_j^* , b_r , and b_r^* and use (13)]:

$$\sum_j [u_{sj} u_{rj}^* - v_{sj} v_{rj}^*] = \delta_{sr} , \quad (21)$$

$$\sum_j [u_{sj} v_{rj} - v_{sj} u_{rj}] = 0 . \quad (22)$$

Now we would like to make sure that the conditions (21)-(22) are not only necessary, but also *sufficient* for the transformation to be canonical. To this end we prove a useful

Lemma. If coefficients u and v satisfy (21)-(22), then the inverse transformation is given by

$$a_j = \sum_s (\tilde{u}_{js} b_s + \tilde{v}_{js} b_s^*) , \quad (23)$$

where

$$\tilde{u}_{js} = u_{sj}^* , \quad \tilde{v}_{js} = -v_{sj} . \quad (24)$$

Problem A6. Prove this lemma by substituting a_j from (23) into the right hand side of (19) .

Now when we know how to express a 's in terms of b 's, we just need to make sure that Eqs. (13) are satisfied. And this is easily seen from (24).

The relations (21)-(22) for the transformation (23) read

$$\sum_s [\tilde{u}_{js} \tilde{u}_{ks}^* - \tilde{v}_{js} \tilde{v}_{ks}^*] = \delta_{jk} , \quad (25)$$

$$\sum_s [\tilde{u}_{js} \tilde{v}_{ks} - \tilde{v}_{js} \tilde{u}_{ks}] = 0 . \quad (26)$$

With (24) these can be rewritten as

$$\sum_s [u_{sj}^* u_{sk} - v_{sj} v_{sk}^*] = \delta_{jk} , \quad (27)$$

$$\sum_s [u_{sj}^* v_{sk} - v_{sj} u_{sk}^*] = 0 . \quad (28)$$

Now we obtain the equations for the u and v coefficients. Eq. (20) implies

$$E_s b_s = \frac{\partial H}{\partial b_s^*} , \quad (29)$$

which becomes non-trivial if H is expressed in terms of a 's. Differentiating both sides with respect to a_j and a_j^* , and taking into account

$$\frac{\partial b_s}{\partial a_j} \equiv u_{sj} , \quad \frac{\partial b_s}{\partial a_j^*} \equiv v_{sj} , \quad (30)$$

we get

$$E_s u_{sj} = \frac{\partial}{\partial a_j} \frac{\partial H}{\partial b_s^*} = \frac{\partial}{\partial b_s^*} \frac{\partial H}{\partial a_j} , \quad (31)$$

$$E_s v_{sj} = \frac{\partial}{\partial a_j^*} \frac{\partial H}{\partial b_s^*} = \frac{\partial}{\partial b_s^*} \frac{\partial H}{\partial a_j^*} . \quad (32)$$

Note that here it is legitimate to change the order of operators $(\partial/\partial b)$'s and $(\partial/\partial a)$'s, because our transformation is linear and, say, $(\partial/\partial b)$'s are just linear combinations of $(\partial/\partial a)$'s:

$$\frac{\partial}{\partial b_s^*} = \sum_j \left[\frac{\partial a_j}{\partial b_s^*} \frac{\partial}{\partial a_j} + \frac{\partial a_j^*}{\partial b_s^*} \frac{\partial}{\partial a_j^*} \right] = \sum_j \left[\tilde{v}_{js} \frac{\partial}{\partial a_j} + \tilde{u}_{js}^* \frac{\partial}{\partial a_j^*} \right] . \quad (33)$$

Calculating the derivatives

$$\frac{\partial H}{\partial a_j} = \sum_i [A_{ij} a_i^* + B_{ij} a_i] , \quad (34)$$

$$\frac{\partial H}{\partial a_j^*} = \sum_i [A_{ji} a_i + B_{ij}^* a_i^*] , \quad (35)$$

and observing that

$$\frac{\partial a_i}{\partial b_s^*} \equiv \tilde{v}_{is} = -v_{si} , \quad \frac{\partial a_i^*}{\partial b_s^*} \equiv \tilde{u}_{is}^* = u_{si} , \quad (36)$$

we finally arrive at the system of equations:

$$\sum_i [A_{ij} u_{si} - B_{ji} v_{si}] = E_s u_{sj} , \quad (37)$$

$$\sum_i [B_{ji}^* u_{si} - A_{ji} v_{si}] = E_s v_{sj} . \quad (38)$$

Using the vector notation

$$|u_s\rangle = (u_{s,1}, u_{s,2}, u_{s,3}, \dots) , \quad |v_s\rangle = (v_{s,1}, v_{s,2}, v_{s,3}, \dots) , \quad (39)$$

and taking into account (18), we write this system as

$$A^* |u_s\rangle - B |v_s\rangle = E_s |u_s\rangle , \quad (40)$$

$$B^* |u_s\rangle - A |v_s\rangle = E_s |v_s\rangle . \quad (41)$$

We can also write this system in terms of \tilde{u} 's and \tilde{v} 's. Introducing the vectors

$$|\tilde{u}_s\rangle = (\tilde{u}_{1,s}, \tilde{u}_{2,s}, \tilde{u}_{3,s}, \dots) , \quad |\tilde{v}_s\rangle = (\tilde{v}_{1,s}, \tilde{v}_{2,s}, \tilde{v}_{3,s}, \dots) , \quad (42)$$

we have

$$A |\tilde{u}_s\rangle + B^* |\tilde{v}_s^*\rangle = E_s |\tilde{u}_s\rangle , \quad (43)$$

$$B |\tilde{u}_s\rangle + A^* |\tilde{v}_s^*\rangle = -E_s |\tilde{v}_s^*\rangle . \quad (44)$$

The energies of the normal modes, E_s 's, arise as the eigenvalues of the problem. The relations (21)-(22), which in the vector notation read

$$\langle u_r | u_s \rangle - \langle v_r | v_s \rangle = \delta_{sr} , \quad (45)$$

$$\langle v_r^* | u_s \rangle - \langle u_r^* | v_s \rangle = 0 , \quad (46)$$

are automatically guaranteed at $E_r \neq E_s$.

Problem A7. Make sure that this is the case.

Eqs. (40)-(41) are invariant with respect to the transformation

$$|u_s\rangle \rightarrow |v_s^*\rangle , \quad |v_s\rangle \rightarrow |u_s^*\rangle , \quad E_s \rightarrow -E_s . \quad (47)$$

This means that each solution has its counterpart of the opposite energy. However, only one solution of the pair is physically relevant. Namely, the one with

$$\langle u_s | u_s \rangle - \langle v_s | v_s \rangle > 0 . \quad (48)$$

For its counterpart we then have

$$\langle u_s | u_s \rangle - \langle v_s | v_s \rangle < 0, \quad (49)$$

which means that it is impossible to normalize it to unity, in contradiction with (45).

Problem A8. Consider the one-mode Hamiltonian

$$H = a^* a + \frac{1}{2} [\gamma a a + \gamma^* a^* a^*]. \quad (50)$$

(a) Show that without a loss of generality (up to a simple canonical transformation) one may choose γ to be real and positive.

(b) Explore the possibility of solving for the dynamics of this model by Bogoliubov transformation.

(c) In the region of γ where Bogoliubov transformation is not helpful, directly solve the equation of motion.

Evolution as a Canonical Transformation

Remarkably, the evolution of a dynamical system can be considered as a canonical transformation. For the given set of variables $\{a_j\}$ we introduce another set $\{b_j\}$, where, by definition, the value of b_j is equal to the value of a_j after some fixed period of evolution, t . First we note that the definition is consistent, because given the particular form of the Hamiltonian, fixed time period t , and particular initial state $\{a_j\}$, we unambiguously fix $\{a_j(t)\}$ by the equations of motion. Hence, each $b_j = a_j(t)$ can be considered as just a function of all variables $\{a_j\}$, the particular form of the function being related to the form of the Hamiltonian and the time period t . Now we make sure that our transformation is canonical. It is enough to do it in the limit $t \rightarrow 0$, because then the case of finite t can be viewed as just a chain of infinitesimal canonical transformations. Assuming that t is arbitrarily small, from the equation of motion we have

$$b_j = a_j - it \frac{\partial H}{\partial a_j^*} + \mathcal{O}(t^2), \quad (51)$$

and, correspondingly,

$$\frac{\partial b_j}{\partial a_k} = \delta_{jk} - it \frac{\partial^2 H}{\partial a_k \partial a_j^*} + \mathcal{O}(t^2), \quad (52)$$

$$\frac{\partial b_j}{\partial a_k^*} = -it \frac{\partial^2 H}{\partial a_k^* \partial a_j^*} + \mathcal{O}(t^2). \quad (53)$$

To check that the relations (13) do really take place (up to the terms $\sim t^2$), we note that

$$\frac{\partial}{\partial b_j} = \frac{\partial}{\partial a_j} + \mathcal{O}(t). \quad (54)$$

Problem A9. Finish the proof.

If the Hamiltonian is bilinear, then the evolution equation is *linear*. Correspondingly, $\{a_j(t)\}$ is related to $\{a_j(0)\}$ by a linear transformation, which can be written in the vector notation, $|a\rangle \equiv (a_1, a_2, a_3, \dots)$, as

$$|a(t)\rangle = U(t) |a(0)\rangle + V(t) |a^*(0)\rangle. \quad (55)$$

Here U and V are some time-dependent matrices the form of which is defined by the Hamiltonian only—not by the initial state $|a(0)\rangle$. The above-proven theorem states that the transformation (55) is canonical. But the linear canonical transformation is nothing else than the Bogoliubov transformation. Hence, the evolution of a system with bilinear Hamiltonian is given by time-dependent Bogoliubov transformation.

Problem A9. Suppose some bilinear Hamiltonian is diagonalizable by the Bogoliubov transformation (19). Express the matrix elements $U_{jk}(t)$ and $V_{jk}(t)$ in the equation (55) in terms of Bogoliubov matrices, u_{sj} and v_{sj} , and eigenvalues, E_s .

Problem A10. In the case of a single-mode system with a bilinear Hamiltonian, Eq. (55) reads

$$a(t) = U(t) a(0) + V(t) a^*(0), \quad (56)$$

where $U(t)$ and $V(t)$ are some functions of time. Find $U(t)$ and $V(t)$ for the Hamiltonian (50) of the Problem A8.

Stability of Equilibrium Solution

By equilibrium solution of the equation of motion we mean the solution which does not evolve in time: $\forall s, \dot{a}_s \equiv 0$. This immediately implies that the equilibrium solution corresponds to such a point, $\{a_s^{(0)}\}$, in the space of variables $\{a_s\}$, where

$$\forall s : \quad \frac{\partial H}{\partial a_s} = \frac{\partial H}{\partial a_s^*} = 0 \quad (\text{at } a_s = a_s^{(0)}). \quad (57)$$

We will also refer to the equilibrium solution as *equilibrium point*. Eq. (57) means that the equilibrium points are the points of extremal behavior of the Hamiltonian. In particular, the previously discussed points of local minima are the equilibrium points.

At the equilibrium point $\{a_s^{(0)}\}$, a natural question arises of what happens if the state of the system is slightly perturbed. The general way of answering this question is to expand the Hamiltonian in the vicinity of the equilibrium in terms of the shifted variables $a_s \rightarrow a_s - a_s^{(0)}$ (assuming that their absolute values are small enough). The resulting Hamiltonian is of the bilinear form (17), with

$$A_{ij} = \frac{\partial^2 H}{\partial a_i \partial a_j^*}, \quad B_{ij} = \frac{\partial^2 H}{\partial a_i \partial a_j} \quad (\text{at } a_s = a_s^{(0)}). \quad (58)$$

Linear terms of the expansion are zero because of Eq. (57) and the constant term is of no interest.

The equation of motion for the Hamiltonian (17) is

$$i \frac{\partial}{\partial t} |a\rangle = A |a\rangle + B^* |a^*\rangle. \quad (59)$$

Since this is a linear equation, its general solution can be written as a sum of elementary solutions (the necessity of the term with complex conjugated ω comes from the term with complex conjugated a)

$$|a\rangle = e^{-i\omega t} |f\rangle + e^{i\omega^* t} |g\rangle. \quad (60)$$

Substituting this into (59) and separating the terms with $e^{-i\omega t}$ and $e^{i\omega^* t}$, which are linear independent at $\text{Re } \omega \neq 0$, we get

$$A |f\rangle + B^* |g^*\rangle = \omega |f\rangle, \quad (61)$$

$$B |f\rangle + A^* |g^*\rangle = -\omega |g^*\rangle. \quad (62)$$

If $\text{Re } \omega = 0$, we write

$$|a\rangle = e^{\lambda t} |f\rangle, \quad (63)$$

and obtain

$$A |f\rangle + B^* |f^*\rangle = i\lambda |f\rangle. \quad (64)$$

The form of the system of equations (61)-(62) is the same as that of (43)-(44). And this is not a coincidence.—In the case when ω 's are real the elementary solutions (60) correspond to the normal modes found by Bogoliubov

transformation. If at least one of the frequencies ω is not real, then the equilibrium state is dynamically unstable with respect to small perturbations. There will be a mode (with $\text{Im } \omega > 0$) that will be increasing exponentially.

But how does one derive the existence of $\text{Im } \omega > 0$ from the fact of existence of a complex ω ? Why is it impossible to have $\text{Im } \omega < 0$ for all complex ω 's? Actually, this follows from the fact that the evolution of the system can be viewed as the Bogoliubov transformation. Suppose we have some complex $\omega = \omega_0 + i\lambda$:

$$|a(t)\rangle = \left\{ e^{-i\omega_0 t} |f\rangle + e^{i\omega_0 t} |g\rangle \right\} e^{\lambda t} . \quad (65)$$

On the other hand, inverting the Bogoliubov transformation (55), we have

$$|a(0)\rangle = \tilde{U} |a(t)\rangle + \tilde{V} |a^*(t)\rangle , \quad (66)$$

where, in accordance with (24),

$$\tilde{U} = U^\dagger , \quad \tilde{V} = -V^T . \quad (67)$$

Combining (65) and (66), we get

$$|a(0)\rangle = \left\{ e^{-i\omega_0 t} \tilde{U} |f\rangle + e^{i\omega_0 t} \tilde{U} |g\rangle + e^{i\omega_0 t} \tilde{V} |f^*\rangle + e^{-i\omega_0 t} \tilde{V} |g^*\rangle \right\} e^{\lambda t} . \quad (68)$$

The left-hand side of Eq. (68) is t -independent, while the right-hand side contains the global exponential factor $e^{\lambda t}$. Hence, either \tilde{U} , or \tilde{V} , or both should contain the compensating factor $e^{-\lambda t}$. But then, by Eq. (67), the same factor should be present in U , or V , or both. Hence, there should exist an elementary solution containing this factor.

Apart from the dynamic instability, when the solution grows exponentially, there can also be a *statistical* instability. In the case of statistical instability, all ω 's are real, but not all of them are positive. The Hamiltonian thus can be diagonalized by the Bogoliubov transformation, and the dynamics within the bilinear Hamiltonian is just the superposition of oscillating normal modes. However, in a typical real situation, when there are higher order terms in the Hamiltonian (say, an interaction with a heat bath) the amplitudes of all the normal modes will be gradually drifting to higher and higher values, since this leads to the increasing entropy without violating energy conservation: The positive contribution to the total energy from the positive- E modes is compensated by negative contribution from the negative- E modes.