1. The results of IQ tests are known to be normally distributed. Suppose that in 2014, the distribution of IQ test scores for persons aged 18-35 years has a variance $\sigma^2 = 225$. A simple random sample of 9 persons take the IQ test. The sample mean score is 115. Calculate the 50%, 75%, 90% and 95% confidence interval estimates of the unknown population mean IQ score.

**Answer:**

<table>
<thead>
<tr>
<th>Interval Type</th>
<th>Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>50% CI</td>
<td>(111.6, 118.4)</td>
</tr>
<tr>
<td>75% CI</td>
<td>(109.2, 120.8)</td>
</tr>
<tr>
<td>90% CI</td>
<td>(106.8, 123.2)</td>
</tr>
<tr>
<td>95% CI</td>
<td>(105.2, 124.8)</td>
</tr>
</tbody>
</table>

**Solution:**

Let the random variable $X = \text{IQ test result}$ assumed normal with:

- $\mu$ unknown
- $\sigma^2 = 225$, known
- $\sigma = 15$, known

A confidence interval estimate of the unknown mean is given by:

$$\text{estimate} \pm \{ \text{confidence coefficient} \} \{ \text{se of estimate} \}$$

where,

- estimate = observed sample mean $= \bar{X} = 115$
- confidence coefficient = $(1 - \alpha/2)$100th percentile Normal$(0,1)$
- se of estimate = standard error of sample mean $= \text{SE}(\bar{X}) = \sqrt{\sigma^2/n}$

$$= \frac{\sqrt{225}}{9}$$
$$= 15 / 3$$
$$= 5$$
For 50% confidence interval estimate:

\[
1 - \alpha = (1 - 0.50) = 0.50 \\
\alpha/2 = 0.50 / 2 = 0.25
\]

Therefore want (1 - .25)100th or 75th percentile of the Normal(0,1) distribution. If you launch the David Lane calculator, this is seen to be = 0.6745

The required confidence interval estimate is thus,

\[
\hat{X} \pm \left\{ \text{confidence coefficient} \right\} \left\{ \text{se of estimate} \right\} = \hat{X} \pm z_{0.25} \sqrt{\frac{\sigma^2}{n}} = 115 \pm \left\{ 0.6745 \right\} \left\{ 5 \right\} = (111.6, 118.4)
\]

For 75% confidence interval estimate:

\[
1 - \alpha = (1 - 0.25) = 0.75 \\
\alpha/2 = 0.25 / 2 = 0.125
\]

Therefore want (1 - .125)100th or 87.5th percentile of the Normal(0,1) distribution. Now if you launch the David Lane Calculator this is seen to be = 1.1505

The required confidence interval estimate is thus,

\[
\hat{X} \pm \left\{ \text{confidence coefficient} \right\} \left\{ \text{se of estimate} \right\} = \hat{X} \pm z_{0.125} \sqrt{\frac{\sigma^2}{n}} = 115 \pm \left\{ 1.1505 \right\} \left\{ 5 \right\} = (109.2, 120.8)
\]

For 90% confidence interval estimate:

\[
1 - \alpha = (1 - 0.10) = 0.90 \\
\alpha/2 = 0.10 / 2 = 0.05
\]

Therefore want (1 - .05)100th or 95th percentile of the Normal(0,1) distribution. Again, using the David Lane calculator, this is seen to be = 1.645

The required confidence interval estimate is thus,

\[
\hat{X} \pm \left\{ \text{confidence coefficient} \right\} \left\{ \text{se of estimate} \right\} = \hat{X} \pm z_{0.05} \sqrt{\frac{\sigma^2}{n}} = 115 \pm \left\{ 1.645 \right\} \left\{ 5 \right\} = (106.8, 123.2)
\]
For 95% confidence interval estimate:

\[ 1 - \alpha = (1 - 0.05) = 0.95 \]
\[ \alpha/2 = 0.05 / 2 = 0.025 \]

Therefore want (1 - .025)100th or 97.5th percentile of the Normal(0,1) distribution. Here the David Lane calculator tells us this value = 1.96

The required confidence interval estimate is thus,

\[ \text{estimate} \pm \text{confidence coefficient} \times \text{se of estimate} \]
\[ = \bar{X} \pm 1.96 \times \sqrt{\frac{\sigma^2}{n}} \]
\[ = 115 \pm 1.96 \times 5 \]
\[ = (105.2, 124.8) \]

2. What trade-offs are involved in reporting one interval estimate over another?

**Answer:**

For a given probability distribution with a known variance and a fixed sample size,

(i) Increasing the confidence coefficient is at the price of a wider confidence interval.

(ii) Decreasing the width of a confidence interval estimate is at the price of a lower confidence coefficient.

This is apparent in the following summary of the solution to Exercise #1.

<table>
<thead>
<tr>
<th>Desired Confidence</th>
<th>Lower Limit</th>
<th>Upper Limit</th>
<th>Width = [Upper limit – Lower limit]</th>
</tr>
</thead>
<tbody>
<tr>
<td>.50</td>
<td>111.6</td>
<td>118.4</td>
<td>6.8</td>
</tr>
<tr>
<td>.75</td>
<td>109.2</td>
<td>120.8</td>
<td>11.6</td>
</tr>
<tr>
<td>.90</td>
<td>106.8</td>
<td>123.2</td>
<td>16.4</td>
</tr>
<tr>
<td>.95</td>
<td>105.2</td>
<td>124.8</td>
<td>19.6</td>
</tr>
</tbody>
</table>

3. If it is known that the population mean IQ score is \( \mu = 105 \) and \( \sigma^2 = 225 \), what proportion of samples of size 6 will result in sample mean values in the interval [135,150]?

**Answer:** << .0001

**Solution:**

Solve for \( \text{Prob} [135 < \bar{X}_{n=6} < 150] \) using the standardization formula.

Note that \( \bar{X}_{n=6} \) is normally distributed with:

\[ \mu_{\bar{X}} = 105 \]
Thus,

\[
\text{Probability } [135 < \bar{X}_{n=6} < 150] = \text{Probability } \left[ \frac{135 - 105}{6.1238} < \frac{\bar{X}_{n=6} - \mu_X}{SE_{\bar{X}}} < \frac{150 - 105}{6.1238} \right]
\]

\[
= \text{Probability } [4.89 < Z\text{-score} < 7.34] << .0001
\]

4. An entomologist samples a field for egg masses of a harmful insect by placing a yard-square frame at random locations and carefully examining the ground within the frame. A simple random sample of 75 locations selected from a county’s pasture land found egg masses in 13 locations. Compute a 95% confidence interval estimate of all possible locations that are infested.

Answer: (.0876, .2590)

Solution:
The setting is estimation of a binomial proportion \( \pi \). In this exercise, the number of trials is \( N=75 \). Since this is sufficiently large, we can obtain a confidence interval using \( \hat{\pi} \pm (z_{1-\alpha/2})S\hat{E}(\hat{\pi}) \) using the standard error formula \“(3)\” that appears on page 60. Thus, the calculations are

\[
\hat{\pi} = \frac{X}{N} = \frac{13}{75} = .1733
\]

\[
S\hat{E} = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{N}} = \sqrt{\frac{(.1733)(.8267)}{75}} = .0437
\]

\[
z_{1-\alpha/2} = z_{.975} = 1.96
\]

\[
\hat{\pi} \pm (z_{1-\alpha/2})S\hat{E}(\hat{\pi}) = .1733 \pm (1.96)(.0437) = (.0876, .2590)
\]