Unit 12
Simple Linear Regression and Correlation

“Assume that a statistical model such as a linear model is a good first start only”

- Gerald van Belle

Is higher blood pressure in the mom associated with a lower birth weight of her baby? Simple linear regression explores the relationship of one continuous outcome ($Y=$birth weight) with one continuous predictor ($X=$blood pressure). At the heart of statistics is the fitting of models to data followed by an examination of how the models perform.

-1- “somewhat useful”
The fitted model is a sufficiently good fit to the data if it permits exploration of hypotheses such as “higher blood pressure during pregnancy is associated with statistically significant lower birth weight” and it permits assessment of confounding, effect modification, and mediation. These are ideas that will be developed in BIOSTATS 640 Unit 2, Multivariable Linear Regression.

-2- “more useful”
The fitted model can be used to predict the outcomes of future observations. For example, we might be interested in predicting the birth weight of the baby born to a mom with systolic blood pressure 145 mm Hg.

-3- “most useful”
Sometimes, but not so much in public health, the fitted model derives from a physical-equation. An example is Michaelis-Menton kinetics. A Michaelis-Menton model is fit to the data for the purpose of estimating the actual rate of a particular chemical reaction.

Hence – “A linear model is a good first start only…”

Nature _________ Population/Sample _________ Observation/Data _________ Relationships/Modeling _________ Analysis/Synthesis
The dangers of extrapolating …

Source: Stack Exchange.
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1. Unit Roadmap

Simple linear regression is used when there is one response (dependent, $Y$) variable and one explanatory (independent, $X$) variables and both are continuous.

Examples of explanatory (independent) – response (dependent) variable pairs are height and weight, age and blood pressure, etc.

-1- A simple linear regression analysis begins with a scatterplot of the data to “see” if a straight line model is appropriate:

$$ y = \beta_0 + \beta_1 x $$

where

$Y$ = the response or dependent variable

$X$ = the explanatory or independent variable.

-2- The sample data are used to estimate the parameter values and their standard errors.

$\beta_1$ = slope (the change in $y$ per 1 unit change in $x$)

$\beta_0$ = intercept (the value of $y$ when $x=0$)

-3- The fitted model is then compared to the simpler model $y = \beta_0$ which says that $y$ is not linearly related to $x$. 

Nature/Populations

Sample

Observation/Data

Relationships/Modeling

Analysis/Synthesis
2. Learning Objectives

When you have finished this unit, you should be able to:

- Explain what is meant by independent versus dependent variable and what is meant by a linear relationship;
- Produce and interpret a scatterplot;
- Define and explain the intercept and slope parameters of a linear relationship;
- Explain the theory of least squares estimation of the intercept and slope parameters of a linear relationship;
- Calculate by hand least squares estimation of the intercept and slope parameters of a linear relationship;
- Explain the theory of the analysis of variance of simple linear regression;
- Calculate by hand the analysis of variance of simple linear regression;
- Explain, compute, and interpret $R^2$ in the context of simple linear regression;
- State and explain the assumptions required for estimation and hypothesis tests in regression;
- Explain, compute, and interpret the overall F-test in simple linear regression;
- Interpret the computer output of a simple linear regression analysis from a package such as R, Stata, SAS, SPSS, Minitab, etc.;
- Define and interpret the value of a Pearson Product Moment Correlation, $r$;
- Explain the relationship between the Pearson product moment correlation $r$ and the linear regression slope parameter; and
- Calculate by hand confidence interval estimation and statistical hypothesis testing of the Pearson product moment correlation $r$. 

Nature | Population/Sample | Observation/Data | Relationships/Modeling | Analysis/Synthesis
3. Definition of the Linear Regression Model

Unit 11 considered two categorical (discrete) variables, such as smoking (yes/no) and low birth weight (yes/no). It was an introduction to chi-square tests of association.

Unit 12 considers two continuous variables, such as age and weight. It is an introduction to simple linear regression and correlation.

A wonderful introduction to the intuition of linear regression can be found in the text by Freedman, Pisani, and Purves (Statistics.  WW Norton & Co., 1978). The following is excerpted from pp 146 and 148 of their text:

“How is weight related to height? For example, there were 411 men aged 18 to 24 in Cycle I of the Health Examination Survey. Their average height was 5 feet 8 inches = 68 inches, with an overall average weight of 158 pounds. But those men who were one inch above average in height had a somewhat higher average weight. Those men who were two inches above average in height had a still higher average weight. And so on. On the average, how much of an increase in weight is associated with each unit increase in height? The best way to get started is to look at the scattergram for these heights and weights. The object is to see how weight depends on height, so height is taken as the independent variable and plotted horizontally …

… The regression line is to a scatter diagram as the average is to a list. The regression line estimates the average value for the dependent variable corresponding to each value of the independent variable.”

Linear Regression

Linear regression models the mean \( \mu = E[Y] \) of one random variable \( Y \) as a linear function of one or more other variables (called predictors or explanatory variables) that are treated as fixed. The estimation and hypothesis testing involved are extensions of ideas and techniques that we have already seen. In linear regression,

- \( Y \) is the outcome or dependent variable that we observe. We observe its values for individuals with various combinations of values of a predictor or explanatory variable \( X \). There may be more than one predictor “\( X \)”; this will be discussed in BIOSTATS 640.

- In simple linear regression the values of the predictor “\( X \)” are assumed to be fixed.

- A simple linear model that relates \( Y \) to one predictor \( X \) is defined as follows:

\[
Y = \beta_0 + \beta_1 X + \text{error}
\]

- It is assumed that at each value of \( X \), the variance of \( Y \) (we call this the conditional variance of \( Y \)) is \( \sigma_{\text{fix}}^2 \)

- Often, however, the variables \( Y \) and \( X \) are both random variables.
Correlation

Correlation considers the association of two random variables.

♦ The techniques of estimation and hypothesis testing are the same for linear regression and correlation analyses.

♦ Exploring the relationship begins with fitting a line to the points.

Development of a simple linear regression model analysis

Example.
Source: Kleinbaum, Kupper, and Muller 1988
The following are observations of age (days) and weight (kg) for n=11 chicken embryos.

<table>
<thead>
<tr>
<th>WT=Y</th>
<th>AGE=X</th>
<th>LOGWT=Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.029</td>
<td>6</td>
<td>-1.538</td>
</tr>
<tr>
<td>0.052</td>
<td>7</td>
<td>-1.284</td>
</tr>
<tr>
<td>0.079</td>
<td>8</td>
<td>-1.102</td>
</tr>
<tr>
<td>0.125</td>
<td>9</td>
<td>-0.903</td>
</tr>
<tr>
<td>0.181</td>
<td>10</td>
<td>-0.742</td>
</tr>
<tr>
<td>0.261</td>
<td>11</td>
<td>-0.583</td>
</tr>
<tr>
<td>0.425</td>
<td>12</td>
<td>-0.372</td>
</tr>
<tr>
<td>0.738</td>
<td>13</td>
<td>-0.132</td>
</tr>
<tr>
<td>1.13</td>
<td>14</td>
<td>0.053</td>
</tr>
<tr>
<td>1.882</td>
<td>15</td>
<td>0.275</td>
</tr>
<tr>
<td>2.812</td>
<td>16</td>
<td>0.449</td>
</tr>
</tbody>
</table>

Notation

♦ The data are 11 pairs of \((X_i, Y_i)\) where \(X=\text{AGE}\) and \(Y=\text{WT}\)
\((X_1, Y_1) = (6, .029) \ldots (X_{11}, Y_{11}) = (16, 2.812)\) and

♦ This table also provides 11 pairs of \((X_i, Z_i)\) where \(X=\text{AGE}\) and \(Z=\text{LOGWT}\)
\((X_1, Z_1) = (6, -1.538) \ldots (X_{11}, Z_{11}) = (16, 0.449)\)
**Research question**

There are a variety of possible research questions:

1. Does weight change with age?

2. In the language of analysis of variance we are asking the following: Can the variability in weight be explained, to a significant extent, by variations in age?

3. What is a “good” functional form that relates age to weight?

**Tip!** Begin with a Scatter plot. Here we plot X=AGE versus Y=WT

We check and learn about the following:

- The average and median of X
- The range and pattern of variability in X
- The average and median of Y
- The range and pattern of variability in Y
- The nature of the relationship between X and Y
- The strength of the relationship between X and Y
- The identification of any points that might be influential
Example, continued

- The plot suggests a relationship between AGE and WT
- A straight line might fit well, but another model might be better
- We have adequate ranges of values for both AGE and WT
- There are no outliers

The “bowl” shape of our scatter plot suggests that perhaps a better model relates the logarithm of WT \((Z=\text{LOGWT})\) to AGE:

![Scatter Plot of LOGWT versus AGE](image)

<table>
<thead>
<tr>
<th>Nature</th>
<th>Population/ Sample</th>
<th>Observation/ Data</th>
<th>Relationships/ Modeling</th>
<th>Analysis/ Synthesis</th>
</tr>
</thead>
</table>
We might have gotten any of a variety of plots.

![Plot 1](image1.png)

No relationship between X and Y

![Plot 2](image2.png)

Linear relationship between X and Y

![Plot 3](image3.png)

Non-linear relationship between X and Y
Note the outlying point

Here, a fit of a linear model will yield an estimated slope that is spuriously non-zero.

Note the outlying point

Here, a fit of a linear model will yield an estimated slope that is spuriously near zero.

Note the outlying point

Here, a fit of a linear model will yield an estimated slope that is spuriously high.
Review of the Straight Line

Way back when, in your high school days, you may have been introduced to the straight line function, defined as “\( y = mx + b \)” where \( m \) is the slope and \( b \) is the intercept. Nothing new here. All we’re doing is changing the notation a bit:

1. **Slope**: \( m \rightarrow \beta_1 \)
2. **Intercept**: \( b \rightarrow \beta_0 \)

\[ y = \beta_0 + \beta_1 x \]

\( \beta_0 = \) “y-intercept” = value of \( y \) when \( x = 0 \)

\( \beta_1 = \) “slope” = \( \Delta y/\Delta x \)

\( \beta_0 = \) “y-intercept” = value of \( y \) when \( x = 0 \)

\( \beta_1 = \) “slope” = \( \Delta y/\Delta x = (\text{change in } y)/(\text{change in } x) \)

**Slope**

<table>
<thead>
<tr>
<th>Slope &gt; 0</th>
<th>Slope = 0</th>
<th>Slope &lt; 0</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
<td><img src="image3.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Nature**  
**Population/Sample**  
**Observation/Data**  
**Relationships/Modeling**  
**Analysis/Synthesis**
### Definition of the Straight Line Model

\[ Y = \beta_0 + \beta_1 X \]

<table>
<thead>
<tr>
<th>Population</th>
<th>Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = \beta_0 + \beta_1 X + \varepsilon )</td>
<td>( Y = \hat{\beta}_0 + \hat{\beta}_1 X + e )</td>
</tr>
</tbody>
</table>

\( \beta_0, \beta_1, \) and \( \varepsilon \) are all unknown!!

\( Y = \beta_0 + \beta_1 X \) is measured with error \( \varepsilon \) defined:

\[ \varepsilon = [Y] - [\beta_0 + \beta_1 X] \]

\( \hat{\beta}_0, \hat{\beta}_1 \) and \( e \) are estimates of \( \beta_0, \beta_1 \) and \( \varepsilon \)

Note: So you know, these may also be written as \( b_0, b_1, \) and \( e \)

**Residual** \( e \) is now the difference between the observed and the fitted (not the true)

\[ e = [Y] - [\hat{\beta}_0 + \hat{\beta}_1 X] \]

**Regression diagnostics are discussed in BIOSTATS 640**

### Notation … sorry …

- \( Y \) = the outcome or dependent variable
- \( X \) = the predictor or independent variable

\[ \mu_Y = \text{The expected value of } Y \text{ for all persons in the population} \]
\[ \mu_{Y|X=x} = \text{The expected value of } Y \text{ for the sub-population for whom } X=x \]

\[ \sigma_Y^2 = \text{Variability of } Y \text{ among all persons in the population} \]
\[ \sigma_{Y|X=x}^2 = \text{Variability of } Y \text{ for the sub-population for whom } X=x \]
4. Estimation

Least squares estimation is used to obtain guesses of \( \beta_0 \) and \( \beta_1 \).

When the outcome = \( Y \) is distributed normal, least squares estimation is the same as maximum likelihood estimation. Note – If you are not familiar with “maximum likelihood estimation”, don’t worry. This is introduced in BIOSTATS 640.

“Least Squares”, “Close” and Least Squares Estimation

Theoretically, it is possible to draw many lines through an X-Y scatter of points. Which to choose? “Least squares” estimation is one approach to choosing a line that is “closest” to the data.

- \( d_i = \text{[observed } Y - \text{ fitted } \hat{Y} \text{]} \) for the \( i^{th} \) person
  Perhaps we’d like \( d_i = \text{[observed } Y - \text{ fitted } \hat{Y} \text{]} = \text{smallest possible.} \)
  Note that this is a vertical distance, since it is a distance on the vertical axis.

- \( d_i^2 = \left[ Y_i - \hat{Y}_i \right]^2 \)
  Better yet, perhaps we’d like to minimize the squared difference:
  \( d_i^2 = \text{[observed } Y - \text{ fitted } \hat{Y} \text{]}^2 = \text{smallest possible} \)

- **Glitch.** We can’t minimize each \( d_i^2 \) separately. In particular, it is not possible to choose common values of \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) that minimizes

\[
\begin{align*}
\sum_{i=1}^{n} d_i^2 &= \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 \quad \text{for subject } 1 \text{ and minimizes} \\
\sum_{i=1}^{n} d_i^2 &= \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 \quad \text{for subject } 2 \text{ and minimizes} \\
\ldots & \quad \ldots \text{ and minimizes} \\
\sum_{i=1}^{n} d_i^2 &= (Y_n - \hat{Y}_n)^2 \quad \text{for the nth subject}
\end{align*}
\]

- So, instead, we choose values for \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) that, upon insertion, minimizes the total

\[
\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} \left[ Y_i - \left( \hat{\beta}_0 + \hat{\beta}_1 X_i \right) \right]^2
\]
For each observed value $x_i$, we have an observed $y_i$, and the “predicted” value $\hat{y}_i$, on the line. The vertical distances $d_i = (y_i - \hat{y}_i)$.

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - [\hat{\beta}_0 + \hat{\beta}_1 x_i])^2$$

has a variety of names:

- residual sum of squares, SSE or SSQ(residual)
- sum of squares about the regression line
- sum of squares due error (SSE)
Least Squares Estimation of the Slope and Intercept

In case you’re interested ….

♦ Consider \( SSE = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} \left( Y_i - \left( \hat{\beta}_0 + \hat{\beta}_1 X_i \right) \right)^2 \)

♦ **Step #1:** Differentiate with respect to \( \hat{\beta}_1 \)
   
   Set derivative equal to 0 and solve for \( \hat{\beta}_1 \).

♦ **Step #2:** Differentiate with respect to \( \hat{\beta}_0 \)
   
   Set derivative equal to 0, insert \( \hat{\beta}_1 \) and solve for \( \hat{\beta}_0 \).

## Least Squares Estimation Solutions

Note – the estimates are denoted either using Greek letters with a caret or with Roman letters

<table>
<thead>
<tr>
<th>Estimate of Slope</th>
<th>( \hat{\beta}_1 ) or ( b_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{\beta}<em>1 = \frac{\sum</em>{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Intercept</th>
<th>( \hat{\beta}_0 ) or ( b_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} )</td>
</tr>
</tbody>
</table>
A closer look …

Some very helpful preliminary calculations

- \( S_{xx} = \sum (X - \bar{X})^2 = \sum X^2 - N\bar{X}^2 \)
- \( S_{yy} = \sum (Y - \bar{Y})^2 = \sum Y^2 - N\bar{Y}^2 \)
- \( S_{xy} = \sum (X - \bar{X})(Y - \bar{Y}) = \sum XY - N\bar{X}\bar{Y} \)

Note - These expressions make use of a “summation notation”, introduced in Unit 1.

The capitol “S” indicates “summation”.
In \( S_{xy} \), the first subscript “x” is saying \((x-x)\).
The second subscript “y” is saying \((y-y)\).

\[
S_{xy} = \sum (X-\bar{X})(Y-\bar{Y})
\]

\( S_y \) subscript \( x \) subscript \( y \)

| Slope          | \[
\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} = \frac{\text{cov}(X,Y)}{\text{var}(X)}
\] | \[
\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}
\]| Intercept       | \[
\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X}
\]| Prediction of \( Y \) | \[
\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1X
\]
|               | = \( b_0 + b_1X \) |
Do these estimates make sense?

| Slope | \[ \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{\text{cov}(X,Y)}{\text{var}(X)} \] | The linear movement in Y with linear movement in X is measured relative to the variability in X. 

\( \hat{\beta}_1 = 0 \) says: 
With a unit change in X, overall there is a 50-50 chance that Y increases versus decreases 

\( \hat{\beta}_1 \neq 0 \) says: 
With a unit increase in X, Y increases also (\( \hat{\beta}_1 > 0 \)) or Y decreases (\( \hat{\beta}_1 < 0 \)). |
| --- | --- | --- |
| Intercept | \[ \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \] | If the linear model is incorrect, or, if the true model does not have a linear component, we obtain 
\( \hat{\beta}_1 = 0 \) and \( \hat{\beta}_0 = \bar{Y} \) as our best guess of an unknown Y |
Illustration in Stata

**Y=WT and X=AGE**

```
regress y x
```

**Partial listing of output ...**

```
+---------------------------------------------+
| y | Coef. | Std. Err. | t    | P>|t| | [95% Conf. Interval] |
|---|-------|-----------|------|-----|----------------------|
| x | .2350727 | .0459425  | 5.12 | 0.001 | .1311437 - .3390018  |
| _cons | -1.884527 | .5258354  | -3.58 | 0.006 | -3.07405 - .695005   |
+---------------------------------------------+
```

**Annotated ...**

```
| y = WEIGHT | Coef. | Std. Err. | t    | P>|t| | [95% Conf. Interval] |
|------------|-------|-----------|------|-----|----------------------|
| x = AGE    | .2350727 = b₁ | .0459425  | 5.12 | 0.001 | .1311437 - .3390018  |
| _cons = Intercept | -1.884527 = b₀ | .5258354  | -3.58 | 0.006 | -3.07405 - .695005   |
```

The fitted line is therefore  \( WT = -1.884527 + 0.23507 \times AGE \). It says that each unit increase in AGE of 1 day is estimated to predict a 0.23507 increase in weight, WT. Here is an overlay of the fitted line on our scatterplot.
As we might have guessed, the straight line model may not be the best choice.

The “bowl” shape of the scatter plot does have a linear component, however.

Without the plot, we might have believed the straight line fit is okay.

Illustration in Stata - continued

Z = LOGWT and X = AGE

```
.regress z x
```

Partial listing of output ...

```
------------------------------------------------------------------------------
|      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval] |
|-------------|----------------------|--------|--------|-----------------------------|
|    x         |   .1958909   .0026768    73.18   0.000     .1898356    .2019462 |
|   _cons     |  -2.689255    .030637   -87.78   0.000     -2.75856    -2.619949 |
------------------------------------------------------------------------------
```

Annotated …

```
Z = LOGWT
|      Coef.       Std. Err.      t    P>|t|     [95% Conf. Interval] |
|-------------|----------------------|--------|--------|-----------------------------|
|    x = AGE  |   .1958909 = b_1  .0026768    73.18   0.000     .1898356    .2019462 |
|_cons = INTERCEPT |  -2.689255 = b_0 .030637   -87.78   0.000     -2.75856    -2.619949 |
------------------------------------------------------------------------------
```

Thus, the fitted line is LOGWT = -2.68925  +  0.19589*AGE
Now the overlay plot looks better:
Now You Try …

**Prediction of Weight from Height**
*Source: Dixon and Massey (1969)*

<table>
<thead>
<tr>
<th>Individual</th>
<th>Height (X)</th>
<th>Weight (Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
<td>110</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>135</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>120</td>
</tr>
<tr>
<td>4</td>
<td>62</td>
<td>120</td>
</tr>
<tr>
<td>5</td>
<td>62</td>
<td>140</td>
</tr>
<tr>
<td>6</td>
<td>62</td>
<td>130</td>
</tr>
<tr>
<td>7</td>
<td>62</td>
<td>135</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>150</td>
</tr>
<tr>
<td>9</td>
<td>64</td>
<td>145</td>
</tr>
<tr>
<td>10</td>
<td>70</td>
<td>170</td>
</tr>
<tr>
<td>11</td>
<td>70</td>
<td>185</td>
</tr>
<tr>
<td>12</td>
<td>70</td>
<td>160</td>
</tr>
</tbody>
</table>

Preliminary calculations

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{X} = 63.833$</td>
<td>$\bar{Y} = 141.667$</td>
</tr>
<tr>
<td>$\sum X_i^2 = 49,068$</td>
<td>$\sum Y_i^2 = 246,100$</td>
</tr>
<tr>
<td>$\sum X_i Y_i = 109,380$</td>
<td>$S_{xx} = 171.667$</td>
</tr>
<tr>
<td>$S_{yy} = 5,266.667$</td>
<td>$S_{xy} = 863.333$</td>
</tr>
</tbody>
</table>

**Slope**

$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$

$\hat{\beta}_1 = \frac{863.333}{171.667} = 5.0291$

**Intercept**

$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

$\hat{\beta}_0 = 141.667 - (5.0291)(63.833) = -179.3573$
5. The Analysis of Variance Table

Recall the sample variance introduced in In Unit 1, *Summarizing Data*.

The numerator of the sample variance \((S^2)\) of the \(Y\) data is \(\sum_{i=1}^{n} (Y_i - \bar{Y})^2\).

This *same* quantity \(\sum_{i=1}^{n} (Y_i - \bar{Y})^2\) is a central figure in regression. It has a new name, several actually.

\[
\sum_{i=1}^{n} (Y_i - \bar{Y})^2 = \text{"total variance of the Y’s".}
\]

\[= \text{"total sum of squares"},\]
\[= \text{"total, corrected"}, \text{ and}\]
\[= \text{"SSY"}.\]

(Nota – “corrected” refers to subtracting the mean before squaring.)

The analysis of variance tables is all about \(\sum_{i=1}^{n} (Y_i - \bar{Y})^2\) and partitioning it into two components

1. **Due residual** (the individual \(Y\) about the individual prediction \(\hat{Y}\))
2. **Due regression** (the prediction \(\hat{Y}\) about the overall mean \(\bar{Y}\))

Here is the partition (Nota – Look closely and you’ll see that both sides are the same)

\[
(Y_i - \bar{Y}) = (Y_i - \hat{Y}_i) + (\hat{Y}_i - \bar{Y})
\]

Some algebra (not shown) reveals a nice partition of the total variability.

\[
\sum (Y_i - \bar{Y})^2 = \sum (Y_i - \hat{Y}_i)^2 + \sum (\hat{Y}_i - \bar{Y})^2
\]

**Total Sum of Squares = Due Error Sum of Squares + Due Model Sum of Squares**

Nature | Population/Sample | Observation/Data | Relationships/Modeling | Analysis/Synthesis
A closer look…

Total Sum of Squares = Due Model Sum of Squares + Due Error Sum of Squares

\[
\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 + \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2
\]

- \((Y_i - \bar{Y})\) = deviation of \(Y_i\) from \(\bar{Y}\) that is to be explained
- \((\hat{Y}_i - \bar{Y})\) = “due model”, “signal”, “systematic”, “due regression”
- \((Y_i - \hat{Y}_i)\) = “due error”, “noise”, or “residual”

We seek to explain the total variability \(\sum_{i=1}^{n}(Y_i - \bar{Y})^2\) with a fitted model:

<table>
<thead>
<tr>
<th>What happens when (\beta_1 \neq 0)?</th>
<th>What happens when (\beta_1 = 0)?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A straight line relationship is helpful</td>
<td>A straight line relationship is not helpful</td>
</tr>
<tr>
<td>Best guess is (\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X)</td>
<td>Best guess is (\hat{Y} = \hat{\beta}_0 = \bar{Y})</td>
</tr>
<tr>
<td>Due model “sum of squares” tends to be LARGE because</td>
<td>Due error “sum of squares” tends to be nearly the TOTAL because</td>
</tr>
</tbody>
</table>

\[
(\hat{Y} - \bar{Y}) = (\hat{\beta}_0 + \hat{\beta}_1 X - \bar{Y})
\]

\[
= \bar{Y} - \hat{\beta}_1 \bar{X} + \hat{\beta}_1 X - \bar{Y}
\]

\[
= \hat{\beta}_1 (X - \bar{X})
\]

Due error “sum of squares” has to be small | Due regression “sum of squares” has to be small |

\[
\Rightarrow \frac{\text{due(model)}}{\text{due(error)}} \text{ will be large}
\]

\[
\Rightarrow \frac{\text{due(model)}}{\text{due(error)}} \text{ will be small}
\]
Partitioning the Total Variance
and all things sum of squares and mean squares

1. The total “pie” is what we are partitioning and it is, simply, the variability in the outcome. Thus, the “total” or “total, corrected” refers to the variability of $Y$ about $\bar{Y}$
   - $\sum_{i=1}^{n} (Y_i - \bar{Y})^2$ is called the “total sum of squares”
   - Degrees of freedom = df = (n-1)
   - Division of the “total sum of squares” by its df yields the “total mean square”

2. One “piece of the pie” what the model explains. The “regression” or “due model” refers to the variability of $\hat{Y}$ about $\bar{Y}$
   - $\sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = \hat{\beta}; \sum_{i=1}^{n} (X_i - \bar{X})^2$ is called the “regression sum of squares”
   - Degrees of freedom = df = 1
   - Division of the “regression sum of squares” by its df yields the “regression mean square” or “model mean square”. It is an example of a variance component.

3. The remaining, “other piece of the pie” what’s left over after we’ve explained what we can with our model.
   The “residual” or “due error” refers to the variability of $Y$ about $\hat{Y}$
   - $\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2$ is called the “residual sum of squares”
   - Degrees of freedom = df = (n-2)
   - Division of the “residual sum of squares” by its df yields the “residual mean square”.

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of Squares A measure of variability</th>
<th>Mean Square = Sum of Squares / df A measure of average/typical/mean variability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression due model</td>
<td>1</td>
<td>SSR = $\sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2$</td>
<td>msq(model) = SSR/1</td>
</tr>
<tr>
<td>Residual due error</td>
<td>(n-2)</td>
<td>SSE = $\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2$</td>
<td>msq(residual) = SSE/(n-2) = $\hat{\sigma}_{Y</td>
</tr>
<tr>
<td>Total, corrected</td>
<td>(n-1)</td>
<td>SST = $\sum_{i=1}^{n} (Y_i - \bar{Y})^2$</td>
<td></td>
</tr>
</tbody>
</table>
Be careful! The question we may ask from an analysis of variance table is a limited one.

Does the fit of the straight line model explain a significant portion of the variability of the individual $Y$ about $\bar{Y}$?

Is this fitted model better than using $\bar{Y}$ alone?

We are NOT asking:

Is the choice of the straight line model correct? nor

Would another functional form be a better choice?

We’ll use a hypothesis test approach (another “proof by contradiction” reasoning just like we did in Unit 7!).

♦ Assume, provisionally, the “nothing is going on” null hypothesis that says $\beta_1 = 0$ ("no linear relationship")

♦ Use least squares estimation to estimate a “closest” line

♦ The analysis of variance table provides a comparison of the due regression mean square to the residual mean square

♦ Where does least squares estimation take us, vis a vis the slope $\beta_1$?
  If $\beta_1 \neq 0$ Then due (regression)/due (residual) will be LARGE
  If $\beta_1 = 0$ Then due (regression)/due (residual) will be SMALL

♦ Our p-value calculation will answer the question:
  If the null hypothesis is true and $\beta_1 = 0$ truly, what were the chances of obtaining a value of due (regression)/due (residual) as larger or larger than that observed?

To calculate “chances of extremeness under some assumed null hypothesis”
we need a null hypothesis probability model!
But did you notice? So far, we have not actually used one!
6. Assumptions for a Straight Line Regression Analysis

In performing least squares estimation, we did not use a probability model. We were doing geometry. Confidence interval estimation and hypothesis testing require some assumptions and a probability model. Here you go!

**Assumptions for Simple Linear Regression**

♦ The separate observations $Y_1, Y_2, \cdots, Y_n$ are independent.

♦ The values of the predictor variable $X$ are fixed and measured without error.

♦ For each value of the predictor variable $X=x$, the distribution of values of $Y$ follows a normal distribution with mean equal to $\mu_{Y|X=x}$ and common variance equal to $\sigma_{Y|x}^2$.

♦ The separate means $\mu_{Y|X=x}$ lie on a straight line; that is –

$$\mu_{Y|X=x} = \beta_0 + \beta_1 X$$

At each value of $X$, there is a population of $Y$ for persons with $X=x$

![Diagram showing assumptions for simple linear regression]

*For each value of $x$, the values of $y$ are normally distributed around $\mu_{y|x}$ on the line, with the same variance for all values of $x$, but different means, $\mu_{y|x}$. Here, $\sigma^2_{y|x_1} = \sigma^2_{y|x_2} = \sigma^2_{y|x_3} = \sigma^2_{y|x_4}$*
With these assumptions, we can assess the significance of the variance explained by the model.

\[
F = \frac{\text{mean square(model)}}{\text{mean square(residual)}} = \frac{\text{msq(model)}}{\text{msq(residual)}} \quad \text{with df = 1, (n-2)}
\]

<table>
<thead>
<tr>
<th>When ( \beta_1 = 0 )</th>
<th>When ( \beta_1 \neq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean square model, msq(model), has expected value ( \sigma_{Y</td>
<td>X}^2 )</td>
</tr>
<tr>
<td>Mean square residual, msq(residual), has expected value ( \sigma_{Y</td>
<td>X}^2 )</td>
</tr>
<tr>
<td>( F = \text{msq(model)/msq(residual)} ) tends to be close to 1</td>
<td>( F = \text{msq(model)/msq(residual)} ) tends to be LARGER than 1</td>
</tr>
</tbody>
</table>

We obtain the analysis of variance table for the model of \( Z=\text{LOGWT} \) to \( X=\text{AGE} \):

Stata illustration with annotations in red.

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>Number of obs = 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>4.22105734</td>
<td>1</td>
<td>4.22105734</td>
<td>( F(1, 9) = 5355.60 ) = MSQ(model)/MSQ(residual)</td>
</tr>
<tr>
<td>Residual</td>
<td>.007093416</td>
<td>9</td>
<td>.000788157</td>
<td>Prob &gt; F = 0.0000 = p-value for Overall F Test</td>
</tr>
<tr>
<td>Total</td>
<td>4.22815076</td>
<td>10</td>
<td>.422815076</td>
<td>R-squared = 0.9983 = SSQ(model)/SSQ(TOTAL)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Adj R-squared = 0.9981 = ( R^2 ) adjusted for n and # of X</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Root MSE = .02807 = Square root of MSQ(residual)</td>
</tr>
</tbody>
</table>
This output corresponds to the following.

Note – In this example our dependent variable is actually $Z$, not $Y$.

<table>
<thead>
<tr>
<th>Source</th>
<th>Df</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression due model</td>
<td>1</td>
<td>$SSR = \sum_{i=1}^{n} (\hat{Z}_i - \bar{Z})^2 = 4.22063$</td>
<td>$msq(model) = SSR/1 = 4.22063$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>You might see $msq(model) = msr$</td>
</tr>
<tr>
<td>Residual due error</td>
<td>(n-2) = 9</td>
<td>$SSE = \sum_{i=1}^{n} (Z_i - \hat{Z}_i)^2 = 0.00705$</td>
<td>$msq(residual) = SSE/(n-2) = 7/838E-04$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>You might see $msq(residual) = mse$</td>
</tr>
<tr>
<td>Total, corrected</td>
<td>(n-1) = 10</td>
<td>$SST = \sum_{i=1}^{n} (Z_i - \bar{Z})^2 = 4.22768$</td>
<td></td>
</tr>
</tbody>
</table>

Other information in this output:

- **R-SQUARED** = $\frac{(\text{Sum of squares regression})}{(\text{Sum of squares total})}$
  = proportion of the “total” that we have been able to explain with the fit
  = “percent of variance explained by the model”

  - *Be careful!* As predictors are added to the model, R-SQUARED can only increase. Eventually, we need to “adjust” this measure to take this into account. See ADJUSTED R-SQUARED.

- We also get an overall F test of the null hypothesis that the simple linear model does not explain significantly more variability in LOGWT than the average LOGWT. $F = \frac{MSQ (\text{Regression})}{MSQ (\text{Residual})}$

  $= \frac{4.22063/0.0007838}{5384.94}$ with df = 1, 9

  p-value = achieved significance < 0.0001. This is a highly unlikely outcome! $\Rightarrow$ Reject $H_0$. Conclude that the fitted line explains statistically significantly more of the variability in $Z=\text{LOGWT}$ than is explained by the intercept-only null hypothesis model.
7. Hypothesis Testing

Straight Line Model: \( Y = \beta_0 + \beta_1 X \)

1) Overall F-Test

**Research Question:** Does the fitted model, the \( \hat{Y} \), explain significantly more of the total variability of the \( Y \) about \( \bar{Y} \) than does \( \bar{Y} \)? A bit of clarification here, in case you’re wondering. When the null hypothesis is true, at least two things happen: (1) \( \beta_1 = 0 \) and (2) the correct model (the null one) says \( Y = \beta_0 + \text{error} \). In this situation, the least squares estimate of \( \beta_0 \) turns out to be \( \bar{Y} \) (that seems reasonable, right?)

**Assumptions:** As before.

**\( H_0 \) and \( H_A \):**

\[
\begin{align*}
H_O: \beta_1 & = 0 \\
H_A: \beta_1 & \neq 0
\end{align*}
\]

**Test Statistic:**

\[
F = \frac{\text{msq(regresion)}}{\text{msq(residual)}}
\]

\[
df = 1, (n - 2)
\]

**Evaluation rule:**

When the null hypothesis is true, the value of \( F \) should be close to 1. Alternatively, when \( \beta_1 \neq 0 \), the value of \( F \) will be LARGER than 1.

Thus, our p-value calculation answers: “What are the chances of obtaining our value of the \( F \) or one that is larger if we believe the null hypothesis that \( \beta_1 = 0 \)?”

**Calculations:**

For our data, we obtain p-value =

\[
\text{pr}\left[F_{1, (n-2)} \geq \left| \frac{\text{msq(model)}}{\text{msq(residual)}} \right| \quad \beta_1=0 \right] = \text{pr}\left[F_{1, 9} \geq 5384.94 \right] < .0001
\]
Evaluate:
Assumption of the null hypothesis that $\beta_1 = 0$ has led to an extremely unlikely outcome (F-statistic value of 5394.94), with chances of being observed less than 1 chance in 10,000. The null hypothesis is rejected.

Interpret:
We have learned that, at least, the fitted straight line model does a much better job of explaining the variability in $Z = \text{LOGWT}$ than a model that allows only for the average LOGWT.

... later ... (BIOSTATS 640, Intermediate Biostatistics), we’ll see that the analysis does not stop here ...

2) Test of the Slope, $\beta_1$

Notes -
The overall F test and the test of the slope are equivalent. The test of the slope uses a t-score approach to hypothesis testing. It can be shown that $\{ \text{t-score for slope} \}^2 = \{ \text{overall F} \}$

Research Question: Is the slope $\beta_1 = 0$?

Assumptions: As before.

$H_O$ and $H_A$:

$H_O : \beta_1 = 0$

$H_A : \beta_1 \neq 0$

Test Statistic:

To compute the t-score, we need an estimate of the standard error of $\hat{\beta}_1$

$$S\hat{E}(\hat{\beta}_1) = \sqrt{\text{msq(residual)} \left[ \frac{1}{\sum_{i=1}^n (X_i - \bar{X})^2} \right]}$$
Our t-score is therefore:

\[
    t - score = \left[ \frac{(\text{observed}) - (\text{expected})}{\text{se}(\text{expected})} \right] = \left[ \frac{(\hat{\beta}) - (0)}{\text{se}(\hat{\beta})} \right]
\]

\[df = (n - 2)\]

We can find this information in our Stata output. Annotations are in red.

Recall what we mean by a t-score:

\[t = 73.38\] says “the estimated slope is estimated to be 73.38 standard error units away from the null hypothesis expected value of zero”.

Check that \( \{t\text{-score}\}^2 = \{Overall F\} \):

\[73.38^2 = 5384.62\] which is close.

Evaluation rule:

When the null hypothesis is true, the value of \( t \) should be close to zero. Alternatively, when \( \beta_1 \neq 0 \), the value of \( t \) will be DIFFERENT from 0.

Here, our p-value calculation answers: “Under the assumption of the null hypothesis that \( \beta_1 = 0 \), what were our chances of obtaining a t-statistic value 73.38 standard error units away from its null hypothesis expected value of zero?”
Calculations:

For our data, we obtain p-value =

\[ 2 \Pr \left[ t_{n-2} \geq \left| \frac{\hat{\beta}_1 - 0}{\hat{s}_e(\hat{\beta}_1)} \right| \right] = 2 \Pr \left[ t \geq 73.38 \right] < 0.0001 \]

Evaluate:

Under the null hypothesis that \( \beta_1 = 0 \), the chances of obtaining a t-score value that is 73.38 or more standard error units away from the expected value of 0 is less than 1 chance in 10,000.

Interpret:

The inference is the same as that for the overall F test. The fitted straight line model does a statistically significantly better job of explaining the variability in LOGWT than the sample mean.

3) Test of the Intercept, \( \beta_0 \)

This addresses the question: Does the straight-line relationship passes through the origin? It is rarely of interest.

Research Question: Is the intercept \( \beta_0 = 0 \)?

Assumptions: As before.

\( H_0 \) and \( H_A \):

\[ H_0 : \beta_0 = 0 \]
\[ H_A : \beta_0 \neq 0 \]
Test Statistic:

To compute the t-score for the intercept, we need an estimate of the standard error of $\hat{\beta}_0$

$$S\hat{E}(\hat{\beta}_0) = \sqrt{msq(residual) \left[ \frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \right]}$$

Our t-score is therefore:

$$t-score = \left[ \frac{(observed) - (expected)}{s\hat{e}(expected)} \right] = \left[ \frac{(\hat{\beta}_0) - (0)}{s\hat{e}(\hat{\beta}_0)} \right]$$

$$df = (n - 2)$$

Again, we can find this information in our Stata output. Annotations are in red.

|       | Coef. | Std. Err. | t = Coef/Std. Err. | P>|t| | [95% Conf. Interval] |
|-------|-------|-----------|--------------------|-----|---------------------|
|       | z     |           |                    |     |                     |
| x     | 0.1958909 | 0.0026768 | 73.18              | 0.000 | 0.1898356    | 0.2019462 |
| _cons | -2.689255 | 0.030637  | -87.78 = -2.689255/0.030637 | 0.000 | -2.75856     | -2.619949 |

Here, $t = -87.78$ says “the estimated intercept is estimated to be 87.78 standard error units away from its null hypothesis expected value of zero”.

Evaluation rule:

When the null hypothesis is true, the value of $t$ should be close to zero. Alternatively, when $\beta_0 \neq 0$, the value of $t$ will be DIFFERENT from 0.

Our p-value calculation answers: “Under the assumption of the null hypothesis that $\beta_0 = 0$, what were our chances of obtaining a t-statistic value 87.78 standard error units away from its null hypothesis expected value of zero”?
Calculations:

\[ p\text{-value} = \]

\[
2pr \left[ t_{(n-2)} \geq \left| \frac{\hat{\beta}_0 - 0}{se(\hat{\beta}_0)} \right| \right] = 2pr \left[ t_9 \geq 87.78 \right] \ll .001
\]

Evaluate:

Under the null hypothesis that the line passes through the origin, that \( \beta_0 = 0 \), the chances of obtaining a t-score value that is 87.78 or more standard error units away from the expected value of 0 is less than 1 chance in 10,000, again prompting statistical rejection of the null hypothesis.

Interpret:

The inference is that there is statistically significant evidence that the straight line relationship between \( Z = \text{LOGWT} \) and \( X = \text{AGE} \) does not pass through the origin.
8. Confidence Interval Estimation

Straight Line Model: \( Y = \beta_0 + \beta_1 X \)

The confidence intervals here have the usual 3 elements (for review, see again Units 8, 9 & 10):

1) Best single guess (estimate)
2) Standard error of the best single guess (SE[estimate])
3) Confidence coefficient: This will be a percentile from the Student t distribution with \( df = n-2 \)

We might want confidence interval estimates of the following 4 parameters:

(1) Slope
(2) Intercept
(3) Mean of subset of population for whom \( X=x_0 \)
(4) Individual response for person for whom \( X=x_0 \)

---

1) SLOPE  
\[ \text{estimate} = \hat{\beta}_1 \]

\[ s\hat{e}(\hat{b}_1) = \sqrt{\text{msq(residual) } \frac{1}{\sum_{i=1}^{n} (X_i - \bar{X})^2}} = \sqrt{\text{(mse) } \frac{1}{\sum_{i=1}^{n} (X_i - \bar{X})^2}} \]

---

2) INTERCEPT  
\[ \text{estimate} = \hat{\beta}_0 \]

\[ s\hat{e}(\hat{b}_0) = \sqrt{\text{msq(residual) } \left[ \frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \right]} = \sqrt{\text{(mse) } \left[ \frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \right]} \]
3) MEAN at $X=x_0$

\[
\text{estimate} = \hat{Y}_{x=x_0} = \hat{\beta}_0 + \hat{\beta}_1 x_0
\]

\[
\text{s.e} = \sqrt{\text{msq(residual)}} \sqrt{\frac{1}{n} \sum (X_i - \bar{X})^2} = \sqrt{\frac{\text{mse}}{n} \sum (X_i - \bar{X})^2}
\]

4) INDIVIDUAL with $X=x_0$

\[
\text{estimate} = \hat{Y}_{x=x_0} = \hat{\beta}_0 + \hat{\beta}_1 x_0
\]

\[
\text{s.e} = \sqrt{\text{msq(residual)}} \sqrt{1 + \frac{1}{n} \sum (X_i - \bar{X})^2} = \sqrt{\frac{\text{mse}}{n} \sum (X_i - \bar{X})^2}
\]

Example, continued

$Z=\text{LOGWT}$ to $X=\text{AGE}$.

**Stata** yielded the following fit:

|       | Coef.  | Std. Err. | $t$   | $P>|t|$ | [95% Conf. Interval] |
|-------|--------|-----------|-------|---------|-----------------------|
| $x$   | .1958909 | .0026768  | 73.18 | 0.000   | .1898356 - .2019462   |
| _cons | -2.689255 | .030637   | -87.78| 0.000   | -2.75856 - -2.619949  |

95% Confidence Interval for the Slope, $\beta_1$

1) Best single guess (estimate) = $\hat{\beta}_1 = 0.19589$

2) Standard error of the best single guess (SE[estimate]) = $se(\hat{\beta}_1) = 0.00268$

3) Confidence coefficient = 97.5\textsuperscript{th} percentile of Student $t = t_{.975, df=9} = 2.26$

95% Confidence Interval for Slope $\beta_1 = \text{Estimate} \pm (\text{confidence coefficient}) \times \text{SE}$

\[= 0.19589 \pm (2.26)(0.00268)\]
\[= (0.1898, 0.2019)\]
95% Confidence Interval for the Intercept, $\beta_0$

|  | Coef. | Std. Err. | t    | P>|t| | [95% Conf. Interval] |
|---|-------|-----------|------|------|---------------------|
| x | 0.1958909 | 0.0026768 | 73.18 | 0.000 | 0.1898356 - 0.2019462 |
| _cons | -2.689255 | 0.030637 | -87.78 | 0.000 | -2.75856 - 2.619949 |

1) Best single guess (estimate) = $\hat{\beta}_0 = -2.68925$

2) Standard error of the best single guess (SE[estimate]) = $se(\hat{\beta}_0) = 0.03064$

3) Confidence coefficient = 97.5th percentile of Student t = $t_{97.5, df=9} = 2.26$

95% Confidence Interval for Slope $\beta_0 = \text{Estimate} \pm (\text{confidence coefficient})*\text{SE}$

$\beta_0 = -2.68925 \pm (2.26)(0.03064)$

$= (-2.7585,-2.6200)$
For the brave ...

Stata Example, continued

Confidence Intervals for MEAN of Z at Each Value of X.

```stata
. * Fit Z to x
. regress z x
  
. * save fitted values xb (this is internal to Stata) to a new variable called zhat
. predict zhat, xb
  
. ** Obtain SE for MEAN of Z at each X (this is internal to Stata) to a new variable called semeanz
. predict semeanz, stdp
  
. ** Obtain confidence coefficient = 97.5th percentile of T on df=9
. generate tmult=invttail(9,.025)
  
. ** Generate lower and upper 95% CI limits for MEAN of Z at Each X
. generate lowmeanz=zhat -tmult*semeanz
. generate highmeanz=zhat+tmult*semeanz
  
. ** Generate lower and upper 95% CI limits for INDIVIDUAL PREDICTED Z at Each X
. generate lowpredictz=zhat-tmult*sempredictz
. generate highpredictz=zhat+tmult*sempredictz
  
. list x z zhat lowmeanz highmeanz, clean
```

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<tr>
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<th>z</th>
<th>zhat</th>
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<th>highmeanz</th>
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<td>16</td>
<td>0.449</td>
<td>0.445</td>
<td>0.4808234</td>
</tr>
</tbody>
</table>

Nature | Population/Sample | Observation/Data | Relationships/Modeling | Analysis/Synthesis
Stata Example, continued

Confidence Intervals for INDIVIDUAL PREDICTED $Z$ at Each Value of $X$.

. * Fit $Z$ to $x$
. regress $z$ $x$

. * Save fitted values to a new variable called $zhat$
. predict $zhat$, xb

. ** Obtain SE for INDIVIDUAL PREDICTION of $Z$ at given $X$ (internal to Stata) to a new variable $sepredictz$
. predict $sepredictz$, stdf

. ** Obtain confidence coefficient = 97.5th percentile of $T$ on df=9
. generate $tmult$=invttail($\nu$,.025)

. ** Generate lower and upper 95% CI limits for INDIVIDUAL PREDICTED $Z$ at Each $X$
. generate $lowpredictz$=$zhat$-$tmult$*$sepredictz$
. generate $highpredictz$=$zhat$+$tmult$*$sepredictz$

. *** List Individual Predictions with 95% CI Limits
. list $x$ $z$ $zhat$ $lowpredictz$ $highpredictz$, clean

<table>
<thead>
<tr>
<th>$x$</th>
<th>$z$</th>
<th>$zhat$</th>
<th>lowpredictz</th>
<th>highpredictz</th>
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<td>.445</td>
<td>.372085</td>
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</table>
9. Introduction to Correlation

Definition of Correlation

A correlation coefficient is a measure of the association between two paired random variables (e.g. height and weight).

The Pearson product moment correlation, in particular, is a measure of the strength of the *straight line* relationship between the two random variables.

Another correlation measure (not discussed here) is the Spearman correlation. It is a measure of the strength of the *monotone increasing (or decreasing)* relationship between the two random variables. The Spearman correlation is a non-parametric (meaning model free) measure. It is introduced in BIOSTATS 640, *Intermediate Biostatistics*.

Formula for the Pearson Product Moment Correlation $\rho$

- Population product moment correlation = $\rho$
- Sample based estimate = $r$
- Some preliminaries:

  (1) Suppose we are interested in the correlation between X and Y

  $\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$ 

  (2) $\text{cov}(X,Y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{(n-1)} = \frac{S_{xy}}{(n-1)}$ \hspace{1cm} This is the covariance(X,Y)

  (3) $\text{var}(X) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{(n-1)} = \frac{S_{xx}}{(n-1)}$ \hspace{1cm} and similarly

  (4) $\text{var}(Y) = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{(n-1)} = \frac{S_{yy}}{(n-1)}$
Formula for Estimate of Pearson Product Moment Correlation from a Sample

\[ \hat{\rho} = r = \frac{\text{cov}(x,y)}{\sqrt{\text{var}(x)\text{var}(y)}} \]

\[ = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} \]

*If you absolutely have to do it by hand, an equivalent (more calculator/excel friendly formula) is*

\[ \hat{\rho} = r = \frac{\sum_{i=1}^{n} x_i y_i - \left( \sum_{i=1}^{n} x_i \right) \left( \sum_{i=1}^{n} y_i \right)}{n \sqrt{\sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2} \sqrt{\sum_{i=1}^{n} y_i^2 - \left( \sum_{i=1}^{n} y_i \right)^2}} \]

- The correlation r can take on values *between 0 and 1 only*
- Thus, the correlation coefficient is said to be *dimensionless* – it is independent of the units of x or y.
- **Sign** of the correlation coefficient (positive or negative) = **Sign** of the estimated slope \( \hat{\beta}_1 \).
There is a relationship between the slope of the straight line, $\hat{\beta}_1$, and the estimated correlation $r$.

| Relationship between slope $\hat{\beta}_1$ and the sample correlation $r$ |
|---|---|
| Because $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$ and $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$ |

A little algebra reveals that

$$r = \frac{\sqrt{S_{xx}}}{\sqrt{S_{yy}}} \hat{\beta}_1$$

**Thus, beware!!!**

- It is possible to have a very large (positive or negative) $r$ might accompanying a very non-zero slope, inasmuch as
  - A very large $r$ might reflect a very large $S_{xx}$, all other things equal
  - A very large $r$ might reflect a very small $S_{yy}$, all other things equal.
10. Hypothesis Test of Correlation

The null hypothesis of zero correlation is equivalent to the null hypothesis of zero slope.

**Research Question:** Is the correlation $\rho = 0$? Is the slope $\beta_1 = 0$?

**Assumptions:** As before.

**$H_0$ and $H_A$:**

\[
H_O : \rho = 0 \\
H_A : \rho \neq 0
\]

**Test Statistic:**
A little algebra (not shown) yields a very nice formula for the t-score that we need.

\[
t-score = \frac{r \sqrt{(n-2)}}{\sqrt{1-r^2}}
\]

\[df = (n-2)\]

We can find this information in our output. Recall the first example and the model of $Z=\text{LOGWT}$ to $X=\text{AGE}$:

The Pearson Correlation, $r$, is the $\sqrt{\text{R-squared}}$ in the output.

\[
\begin{array}{llll}
\text{Source} & \text{SS} & \text{df} & \text{MS} \\
\hline
\text{Model} & 4.22105734 & 1 & 4.22105734 \\
\text{Residual} & .007093416 & 9 & .000788157 \\
\hline
\text{Total} & 4.22815076 & 10 & .422815076 \\
\hline
\end{array}
\]

Number of obs = 11

$F( 1, 9) = 5355.60$

Prob > $F = 0.0000$

R-squared = 0.9983

Adj R-squared = 0.9981

Root MSE = .02807

Pearson Correlation, $r = \sqrt{0.9983} = 0.9991$
Substitution into the formula for the t-score yields

\[
\begin{align*}
\text{t-score} &= \left[ \frac{r \sqrt{n-2}}{\sqrt{1-r^2}} \right] = \left[ \frac{.9991\sqrt{9}}{\sqrt{1-.9983}} \right] = \left[ \frac{2.9974}{.0412} \right] = 72.69 \\
\end{align*}
\]

Note: The value .9991 in the numerator is \( r = \sqrt{R^2} = \sqrt{.9983} = .9991 \)

This is very close to the value of the t-score that was obtained for testing the null hypothesis of zero slope. The discrepancy is probably rounding error. I did the calculations on my calculator using 4 significant digits. Stata probably used more significant digits - cb.