Proof of the Angle Change Invariant Model

Let $u_{ax}$ and $u_{ay}$ be the velocity of Ball A before the collision along the Collision and Sweep Axes, respectively. Let $v_{ax}$ and $v_{ay}$ be the velocity of Ball A after the collision along the Collision and Sweep Axes, respectively. Likewise, let $u_{bx}$, $u_{by}$, $v_{bx}$, and $v_{by}$ be the four velocities for Ball B. Let $u_{mx}$, $u_{my}$, $v_{mx}$, and $v_{my}$ be the average velocities of the two balls along the Collision and Sweep Axis before and after the collision. That is,

$$\begin{aligned} u_{mx} &= \frac{1}{2}(u_{ax} + u_{bx}), \\ u_{my} &= \frac{1}{2}(u_{ay} + u_{by}), \\ v_{mx} &= \frac{1}{2}(v_{ax} + v_{bx}), \\ v_{mx} &= \frac{1}{2}(v_{ay} + v_{by}). \end{aligned} \qquad \text{(A1)}$$

Assuming perfectly elastic balls we know that $u_{my} = v_{my}$ because there is no change of velocity along the Sweep Axis. Thus, only the velocities along the Collision Axis, $u_{mx}$ and $v_{mx}$ are not constrained to be equal.

Given the pre-collision velocities, the post-collision velocities can be calculated as (e.g., Halliday et al, 1993)

$$v_{ax} = \frac{m_a - m_b}{m_a + m_b} u_{ax} + \frac{2m_b}{m_a + m_b} u_{bx} \qquad \text{(A2)}$$

and

$$v_{bx} = \frac{2m_a}{m_a + m_b} u_{ax} + \frac{m_b - m_a}{m_a + m_b} u_{bx}\,, \qquad \text{(A3)}$$

where $m_a$ and $m_b$ are the masses of Balls A and B, respectively. Similar equations can be constructed for $u_{ax}$ and $u_{bx}$.

From Equations in A2 and A3, we know that

$$
\begin{aligned}
v_{mx} &= \frac{1}{2}\left(v_{ax} + v_{bx}\right) \\
&= \frac{1}{2}\left[\left(\frac{m_a - m_b}{m_a + m_b}u_{ax} + \frac{2m_b}{m_a + m_b}u_{bx}\right) + \left(\frac{2m_a}{m_a + m_b}u_{ax} + \frac{m_b - m_a}{m_a + m_b}u_{bx}\right)\right] \\
&= \frac{1}{2}\left[\frac{3m_a - m_b}{m_a + m_b}u_{ax} + \frac{3m_b - m_a}{m_a + m_b}u_{bx}\right],
\end{aligned} \tag{A4}
$$

and similarly for $u_{mx}$. We are interested in the change in the average Collision Axis velocity. That is,

$$
\begin{aligned}
\delta = u_{mx} - v_{mx} &= \frac{1}{2}\left[u_{ax}\left(1 - \frac{3m_a - m_b}{m_a + m_b}\right) + u_{bx}\left(1 - \frac{3m_b - m_a}{m_a + m_b}\right)\right] \\
&= \frac{1}{2}\left[\frac{2m_b - 2m_a}{m_a + m_b}u_{ax} + \frac{2m_a - 2m_b}{m_a + m_b}u_{bx}\right] \\
&= \frac{m_b - m_a}{m_a + m_b}u_{ax} + \frac{m_a - m_b}{m_a + m_b}u_{bx} \\
&= \frac{m_b - m_a}{m_a + m_b}\left(u_{ax} - u_{bx}\right)
\end{aligned} \tag{A5}
$$

For a collision to occur $u_{ax} - u_{bx}$ must be positive. Thus, we have that,

$$
\begin{aligned}
\delta &= 0 \text{ if } m_a = m_b, \\
\delta &> 0 \text{ if } m_a < m_b, \\
\delta &< 0 \text{ if } m_a > m_b.
\end{aligned} \tag{A6}
$$

For a linear collision, if the average Collision Axis velocity increases, Ball A is heavier, if it decreases, Ball B is heavier, and if it remains unchanged, the balls have equal mass. For a collision with a non-zero Sweep Axis component that does not change after the collision, $\alpha$ from Figure 4 is positive if Ball B is heavier and negative if Ball A is heavier.